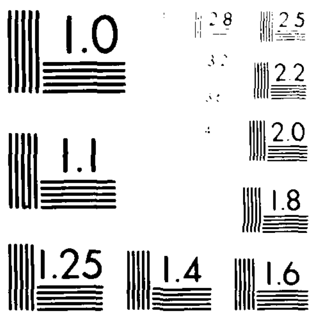
AD-A116 324 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/6 12/1
GLOBAL EXISTENCE OF SOLUTIONS OF THE EQUATIONS OF ONE-DIMENSIONAL
APR 82 J U KIM DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2364 NL

1 OF 1
AD A
116324

END
DATE
FILMED
07-82
DTIC



MICROCOPY RESOLUTION TEST CHART

MRC Technical Summary Report #2364

GLOBAL EXISTENCE OF SOLUTIONS OF THE EQUATIONS OF ONE-DIMENSIONAL THERMOVISCOELASTICITY WITH INITIAL DATA IN BV AND L^1

Jong Uhn Kim

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

April 1982

(Received January 22, 1982)

Sponsored by

U.S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

GLOBAL EXISTENCE OF SOLUTIONS OF THE EQUATIONS OF ONE-DIMENSIONAL THERMOVISCOELASTICITY WITH INITIAL DATA IN BV AND L^1

Jong Uhn Kim

Technical Summary Report #2364

April 1982

ABSTRACT

We consider the Cauchy problem associated with the equations:

$$(1)\quad \begin{cases} u_t = v_x \\ v_t = -p(u,\theta)_x + v_{xx} \\ [e(u,\theta) + \frac{1}{2} v^2]_t + [p(u,\theta)v]_x - [vv_x]_x = \theta_{xx}, \; x \in R, \; t \in R^+ \end{cases} ,$$

with the initial condition

$$(2)\qquad u(0,x) = u_0(x),\; v(0,x) = v_0(x),\; \theta(0,x) = \theta_0(x) \quad .$$

The equations (1) describe the one-dimensional motion of a particular type of nonlinear thermoviscoelastic material. We establish the existence of global solutions when the initial data belong to $L^1 \cap BV$ and are sufficiently small in terms of $L^1 \cap BV$. Our method consists of linearization, Fourier transformations and contraction mapping principle via variation of constants formula.

AMS (MOS) Subject Classifications: 35B99, 35K55, 35M05, 73B99

Key Words: Equations of one-dimensional nonlinear thermoviscoelasticity, Linear equations, Functions of bounded variation (BV), Fourier transform, Global solutions, Variation of constants formula.

Work Unit Number 1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 1.

SIGNIFICANCE AND EXPLANATION

This paper discusses the Cauchy problem associated with a particular system of equations of one-dimensional nonlinear thermoviscoelasticity with the initial data given in the class of functions of bounded variation (denoted by BV). It has been known that the class of BV is a suitable function space for the study of evolution equations which arise in continuum mechanics in order to admit solutions possessing shocks. This fact has been exploited in the analysis of hyperbolic conservation laws which describe the motion of a continuum when mechanical and thermal dissipations are neglected. On the other hand, only the smooth (classical) solutions have been studied for the equations which include such dissipative terms. Our goal is to study the global existence of weaker solutions of systems which include such dissipative terms. Our main result shows that when the initial data are sufficiently small in the L^1 and BV norms, the system (1) of the abstract has global solutions in time possessing specific regularity properties.

A

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

GLOBAL EXISTENCE OF SOLUTIONS OF THE EQUATIONS
OF ONE-DIMENSIONAL THERMOVISCOELASTICITY
WITH INITIAL DATA IN BV AND L^1

Jong Uhn Kim

0. Introduction

The purpose of this paper is to establish existence of global solutions in BV for the Cauchy problem associated with the equations of one-dimensional nonlinear thermoviscoelasticity:

$$(0.1)\qquad \begin{cases} u_t = v_x \\ v_t = -\tilde{p}(u,\theta)_x + v_{xx} \\ [\tilde{e}(u,\theta) + \frac{1}{2} v^2]_t + [\tilde{p}(u,\theta)v]_x - [vv_x]_x = \theta_{xx} \end{cases},$$

$$-\infty < x < \infty, \; 0 \leq t < \infty$$

with initial conditions

$$(0.2)\qquad u(0,x) = \tilde{u}_0(x),\; v(0,x) = \tilde{v}_0(x),\; \theta(0,x) = \tilde{\theta}_0(x) \;,$$

where u, v, θ, $\tilde{p}$ and $\tilde{e}$ denote deformation gradient, velocity, temperature, stress and internal energy, respectively, and the conventional notations for partial derivatives are employed. Equations (0.1) are the conservation laws in Lagrangian form of mass, linear momentum and energy. From physical considerations, we should require the following conditions:

$$(0.3)\qquad \begin{cases} u > 0,\; \theta > 0 \\ \tilde{p}_u(u,\theta) < 0,\; \tilde{e}_\theta(u,\theta) > 0,\; \tilde{e}_u(u,\theta) = \theta^2 [\frac{\tilde{p}(u,\theta)}{\theta}]_\theta \end{cases}.$$

For a detailed account of physical meaning of (0.1), (0.3), the reader is referred to [3], [4].

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 1.

Now let us discuss briefly the significance of our problem. Equations (0.1) have both mechanical and thermal dissipations which preserve the smoothness of initial data. This fact was shown in [3], [4], which treated equations more general than (0.1). Slemrod [7] proved that the thermal dissipation alone is enough to establish the existence of global smooth solutions for initial-boundary value problem with small, smooth initial data. Without dissipation terms, (0.1) reduces to the hyperbolic conservation laws:

$$(0.4) \qquad \begin{cases} u_t = v_x \\ v_t = -\tilde{p}(u,\theta)_x \\ [\tilde{e}(u,\theta) + \frac{1}{2} v^2]_t + [\tilde{p}(u,\theta)v]_x = 0 \quad , \end{cases}$$

which are certainly incapable of smoothing out rough initial data. Nevertheless, the Cauchy problem for (0.4) has global solutions of class BV when the initial data have small variation [5]. We are naturally led to believe that the same is true of any new system of equations obtained from (0.4) by adding dissipation terms [2]. This conjecture has not been verified. In this note, we shall give an affirmative answer with some reasonable assumptions. The main result is Theorem 2.1.

Next we shall give a sketch of our method. For convenience, we introduce new variables $u(t,x) - \bar{u}$, $\theta(t,x) - \bar{\theta}$, which we shall still call by $u(t,x)$ and $\theta(t,x)$, where $\bar{u}$ and $\bar{\theta}$ are positive constants and $(\bar{u},0,\bar{\theta})$ is regarded as the given equilibrium state. At the same time, we define

$$(0.5) \qquad p(u,\theta) = \tilde{p}(\bar{u}+u,\ \bar{\theta}+\theta),\ e(u,\theta) = \tilde{e}(\bar{u}+u,\ \bar{\theta}+\theta) \quad .$$

Then (0.1), (0.2) are equivalent to

$$
(0.6)\qquad \begin{cases} u_t = v_x \\ v_t = -p(u,\theta)_x + v_{xx} \\ [e(u,\theta) + \frac{1}{2} v^2]_t + [p(u,\theta)v]_x - [vv_x]_x = \theta_{xx} \end{cases}
$$

with initial conditions

$$
(0.7)\qquad \begin{aligned} &u(0,x) = u_0(x) \overset{\text{def}}{=} \tilde{u}_0(x) - \bar{u},\ v(0,x) = v_0(x) \overset{\text{def}}{=} \tilde{v}_0(x),\ \theta(0,x) = \\ &= \theta_0(x) \overset{\text{def}}{=} \tilde{\theta}_0(x) - \bar{\theta} \quad , \end{aligned}
$$

while (0.3) is equivalent to

$$
(0.8)\qquad u > -\bar{u},\ \theta > -\bar{\theta} \quad ,
$$

$$
(0.9)\qquad p_u(u,\theta) < 0,\ e_\theta(u,\theta) > 0,\ e_u(u,\theta) = (\bar{\theta}+\theta)p_\theta(u,\theta) - p(u,\theta) \quad .
$$

In addition to these physical assumptions, we assume

(0.10) $p(u,\theta)$, $e(u,\theta)$ are analytic functions of (u,θ) in a neighborhood of $(0,0)$ with $p(0,0) = 0$ and $p_\theta(0,0) \neq 0$.

Assuming everything is smooth enough, (0.6) combined with (0.9) is equivalent to

$$
(0.11)\qquad \begin{cases} u_t = v_x \\ v_t = -p(u,\theta)_x + v_{xx} \\ \theta_t = -\dfrac{p_\theta(u,\theta)}{e_\theta(u,\theta)} (\bar{\theta}+\theta)v_x + \dfrac{1}{e_\theta(u,\theta)} v_x^2 + \dfrac{1}{e_\theta(u,\theta)} \theta_{xx} \quad . \end{cases}
$$

The linearized equations associated with (0.11) are

$$
(0.12)\qquad \begin{cases} u_t = v_x \\ v_t = au_x + b\theta_x + v_{xx} \\ \theta_t = dv_x + c\theta_{xx} \quad , \end{cases}
$$

where a, b, c, d are constants. Now we are in a position to summarize our strategy. First, by the method of Fourier transform, we analyze solutions to (0.12), (0.7), assuming $(u_0,v_0,\theta_0) \in (L^1 \cap BV)^3$. Then we collect all information on the regularity and the asymptotic behavior of solutions to this linear problem. Based on this information, we construct a suitable function space and also a contraction mapping via variation of constants formula so that the fixed point may be the solution to (0.11), (0.7). Finally, we verify that this solution is also a solution to (0.6), (0.7), (0.8) in the same function space. In fact, this approach was used in [6].

As a final remark, it is reported that our method does not work out in case $(u_0,v_0,\theta_0) \in (BV)^3$ rather than $(u_0,v_0,\theta_0) \in (L^1 \cap BV)^3$.

Notation

We use the following notations throughout this paper.

(1) For $f : R^+ \times R \to R$, we write

$$\partial_t f(t,x) = f_t(t,x) = \frac{\partial f}{\partial t}(t,x), \quad \partial_x f(t,x) = f_x(t,x) = \frac{\partial f(t,x)}{\partial x}$$

$$\partial_{xx} f(t,x) = f_{xx}(t,x) = \frac{\partial^2 f(t,x)}{\partial x^2} .$$

(2) For $f \in L^1(R)$, we write $\|f\| = \int_{-\infty}^{\infty} |f(x)|dx$. we adopt the conventional notation for other L^p-norms.

(3) $C_0(R)$ is the space of continuous functions tending to zero at infinity and its dual is denoted by

$\mathcal{M}$: the Banach space of all finite measures.

(4) For $f \in \mathcal{M}$, $\|f\|$ = total variation of f as a measure. Since L^1 is isometrically embedded into $\mathcal{M}$, there is no ambiguity in notation.

(5) $\rho_\varepsilon(x)$ stands for the Friedrichs mollifier.

(6) Convolution is taken with respect to x variable alone unless specified otherwise, and we write

$$f(x)*g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy \quad ,$$

$$\int_0^t f(t-\tau,x)*g(\tau,x)d\tau = \int_0^t \int_{-\infty}^{\infty} f(t-\tau,x-y)g(\tau,y)dyd\tau \quad .$$

(7) F_x means the Fourier transform with respect to x and F_ξ^{-1} means the inverse Fourier transform with respect to ξ. We write $\hat{f}(\xi) = F_x f(x)$ and $f(x) = F_\xi^{-1}\hat{f}(\xi)$.

(8) $\mathcal{D}^*(\Omega)$ stands for the space of all distributions in Ω, where Ω is an open subset of R^n. When X is a Banach space, $\mathcal{D}^*((0,\infty);X)$ denotes the space of X-valued distributions in $(0,\infty)$.

(9) $\Lambda_\beta^{1,\infty}$ is the space of all function f in $L^1(R)$ for which the norm

$$\|f\| + \sup_{h\neq 0} \frac{\|f(x+h)-f(x)\|}{|h|^\beta}$$

is finite, where $0 < \beta < 1$ (see [8]).

(10) For $f \in \Lambda_\beta^{1,\infty}$, we write $|||f|||_\beta = \sup_{h\neq 0} \frac{\|f(x+h)-f(x)\|}{|h|^\beta}$.

(11) The same letter M will be used for different constants which are independent of t. Its independence of other constants will be indicated whenever necessary.

(12) $W^{1,1}$ is the space of all function f in $L^1(R)$ such that $\frac{df}{dx} \in L^1(R)$.

1. Linearized Equations

As stated in the introduction, we shall use the method of Fourier transform to estimate the fundamental solution of the linear equations:

$$(1.1)\qquad \begin{cases} u_t = v_x \\ v_t = au_x + b\theta_x + v_{xx} \\ \theta_t = dv_x + c\theta_{xx} \end{cases}$$

where $a = -p_u(0,0) > 0$, $b = -p_\theta(0,0) \neq 0$, $c = \frac{1}{e_\theta(0,0)} > 0$ and $d = -\frac{p_\theta(0,0)}{e_\theta(0,0)}\bar{\theta}$. Applying the Fourier transform with respect to x, (1.1) yields

$$(1.2)\qquad \frac{\partial}{\partial t}\hat{Y}(t,\xi) = \hat{A}(\xi)\hat{Y}(t,\xi) \quad ,$$

where $\hat{Y}(t,\xi) = \begin{pmatrix} \hat{u}(t,\xi) \\ \hat{v}(t,\xi) \\ \hat{\theta}(t,\xi) \end{pmatrix}$ and $\hat{A}(\xi) = \begin{pmatrix} 0 , & i\xi , & 0 \\ ia\xi , & -\xi^2 , & ib\xi \\ 0 , & id\xi , & -c\xi^2 \end{pmatrix}$.

Denote $e^{t\hat{A}(\xi)}$ by $\hat{G}(t,\xi)$ and $F_\xi^{-1}e^{t\hat{A}(\xi)}$ by $G(t,x)$. We call each entry of the matrix $G(t,x)$ by $G_{ij}(t,x)$, $i,j = 1,2,3$. Our principal objective in this section is to analyze $G_{ij}(t,x)$. Since it is not easy to obtain the explicit formula for $\hat{G}(t,\xi)$, we shall use the Dunford integral to express $\hat{G}(t,\xi)$:

$$(1.3)\qquad e^{t\hat{A}(\xi)} = \frac{1}{2\pi i}\int_\Gamma (\lambda I-\hat{A}(\xi))^{-1}e^{\lambda t}d\lambda$$

where Γ is a contour encircling all the spectrum of $\hat{A}(\xi)$ in the complex plane. This is useful because we know the explicit formula for the integrand. Let us define

$$(1.4)\qquad p(\xi,\lambda) = \lambda^3 + (c+1)\xi^2\lambda^2 + (c\xi^4 + a\xi^2 + bd\xi^2)\lambda + ac\xi^4 \quad .$$

Then, $(\lambda I-\hat{A}(\xi))^{-1}$ is the matrix:

$$\begin{bmatrix} C_{11}, & C_{12}, & C_{13} \\ C_{21}, & C_{22}, & C_{23} \\ C_{31}, & C_{32}, & C_{33} \end{bmatrix} ,$$

where
$$\begin{aligned}
C_{11} &= \{\lambda^2 + (c+1)\xi^2\lambda + bd\xi^2 + c\xi^4\}p(\xi,\lambda)^{-1} , \\
C_{12} &= \{i\xi\lambda + ic\xi^3\}p(\xi,\lambda)^{-1} , \\
C_{13} &= -b\xi^2 p(\xi,\lambda)^{-1} , \\
C_{21} &= \{ia\xi\lambda + iac\xi^3\}p(\xi,\lambda)^{-1} , \\
C_{22} &= \{\lambda^2 + c\xi^2\lambda\}p(\xi,\lambda)^{-1} , \\
C_{23} &= ib\xi\lambda\ p(\xi,\lambda)^{-1} , \\
C_{31} &= -ad\xi^2 p(\xi,\lambda)^{-1} , \\
C_{32} &= id\xi\lambda\ p(\xi,\lambda)^{-1} , \\
C_{33} &= \{\lambda^2 + \xi^2\lambda + a\xi^2\}p(\xi,\lambda)^{-1} .
\end{aligned}$$

(1.3) implies

(1.5) $$\hat{G}_{ij}(t,\xi) = \frac{1}{2\pi i}\int_\Gamma C_{ij}e^{\lambda t}d\lambda, \quad \text{for} \quad i,j = 1,2,3 .$$

It is interesting to see that

(1.6) $$G_{21}(t,x) = aG_{12}(t,x),\ G_{13}(t,x) = \frac{b}{ad}G_{31}(t,x),\ G_{23}(t,x) = \frac{b}{d}G_{32}(t,x) ,$$

which are obvious from the expressions for C_{ij}'s, and that

(1.7) $$\frac{\partial}{\partial t}\begin{bmatrix} G_{11}(t,x), & G_{12}(t,x), & G_{13}(t,x) \\ G_{21}(t,x), & G_{22}(t,x), & G_{23}(t,x) \\ G_{31}(t,x), & G_{32}(t,x), & G_{33}(t,x) \end{bmatrix} =$$

$$= \begin{bmatrix} 0, & \partial_x, & 0 \\ a\partial x, & \partial_{xx}, & b\partial_x \\ 0, & d\partial_x, & c\partial_{xx} \end{bmatrix} \begin{bmatrix} G_{11}(t,x), & G_{12}(t,x), & G_{13}(t,x) \\ G_{21}(t,x), & G_{22}(t,x), & G_{23}(t,x) \\ G_{31}(t,x), & G_{32}(t,x), & G_{33}(t,x) \end{bmatrix}$$

holds in $\mathcal{D}^*((0,\infty) \times R)$.

Before estimating L^1-norm or total variation of each G_{ij} and its derivatives, we shall explain the general strategy of estimtion. First, we analyze the roots of the polynomial equation $p(\xi,\lambda) = 0$, which are the poles of C_{ij}. Second, noting that the value of integral in (1.5) is simply the sum of residues of $C_{ij}e^{\lambda t}$ at each pole, we obtain the residues in the form of infinite series in ξ. Finally, we use the following fact to obtain an estimate of L^1-norm of a function.

<u>Lemma 1.1</u>. Suppose $f(x) \in C_0^\infty(R)$. Then for $0 \leq \beta < \frac{1}{2}$,

$$(1.8) \qquad \| \, |x|^\beta f(x) \| \leq \sqrt{\frac{2}{1+2\beta}}\, T^{\frac{1}{2}+\beta} \|\hat{f}(\xi)\|_{L^2} + \sqrt{\frac{2}{1-2\beta}}\, T^{-\frac{1}{2}+\beta} \|\frac{d}{d\xi}\hat{f}(\xi)\|_{L^2}$$

and

$$(1.9) \qquad \| \, |x|^\beta f(x) \| \leq \frac{2}{1+\beta}\, T^{1+\beta} \|\hat{f}(\xi)\| + \sqrt{\frac{2}{1-2\beta}}\, T^{-\frac{1}{2}+\beta} \|\frac{d}{d\xi}\hat{f}(\xi)\|_{L^2}$$

hold for all $T > 0$.

<u>Proof</u>. The result follows from the inequality

$$\int_{-\infty}^{\infty} |x|^\beta |f(x)|\,dx \leq \int_{|x|\leq T} |x|^\beta |f(x)|\,dx + \int_{|x|\geq T} |x|^{-1+\beta} |xf(x)|\,dx$$

and Hölder's inequality.

According to the theory of algebraic functions [1], the roots of algebraic equations are expressed by the Puiseux series in the parameter in a neighborhood of the multiple root. But for the equation $p(\xi,\lambda) = 0$, it is easy to see that the Puiseux series reduce to the Laurent series in ξ for $|\xi|$ large and to the Taylor series in ξ for $|\xi|$ small.

<u>Lemma 1.2</u>. There exist positive numbers $\tilde{\rho} < \tilde{\eta}$ such that the roots of $P(\xi,\lambda) = 0$ are given by

$$\lambda_1 = i\sqrt{a+bd}\ \xi - \frac{a+bd(c+1)}{2(a+bd)}\ \xi^2 + 0(\xi^3) \quad ,$$

$$\lambda_2 = -i\sqrt{a+bd}\ \xi - \frac{a+bd(c+1)}{2(a+bd)}\ \xi^2 + 0(\xi^3) \quad ,$$

$$\lambda_3 = - \frac{ac}{(a+bd)}\ \xi^2 + 0(\xi^4)$$

if $|\xi| < \tilde{\rho}$ and

$$\tilde{\lambda}_1 = -c\xi^2 + \frac{bd}{c-1} + 0(\frac{1}{\xi^2}) \quad ,$$

$$\tilde{\lambda}_2 = -\xi^2 + \frac{ac-a-bd}{c-1} + 0(\frac{1}{\xi^2}) \quad ,$$

$$\tilde{\lambda}_3 = -a + \frac{abd-a^2c}{c}\ \frac{1}{\xi^2} + 0(\frac{1}{\xi^4})$$

if $|\xi| > \tilde{\eta}$, where the standard symbol $0(\cdot)$ denotes the remainder of the Taylor or Laurent series.

We omit the proof which can be given by direct computation.

Remark 1.3. In stating above lemma, it was implicitly assumed that $c \neq 1$. The analysis for the case $c = 1$ may be a little different from the technical viewpoint. But the estimates for $G_{ij}(t,x)$ are the same and we assume $c \neq 1$ throughout this paper.

Lemma 1.4. $\hat{G}_{ij}(t,\xi)$'s, $i,j = 1,2,3$, are analytic functions of ξ for each $t > 0$ and they can be expressed in the following forms: If $|\xi| < \rho < \tilde{\rho}$,

$$(1.10)\quad \hat{G}_{11}(t,\xi) = \frac{bd+0(\xi^2)}{a+bd+0(\xi)}\ e^{t\{-\frac{ac}{a+bd}\xi^2 + 0(\xi^4)\}}$$

$$+ \frac{a+0(\xi)}{2(a+bd)+0(\xi)}\ e^{t\{-i\sqrt{a+bd}\ \xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

$$+ \frac{a+0(\xi)}{2(a+bd)+0(\xi)}\ e^{t\{i\sqrt{a+bd}\ \xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

$$(1.11)\quad \hat{G}_{12}(t,\xi) = \frac{(ic-\frac{iac}{a+bd})\,\xi + 0(\xi^3)}{a+bd + 0(\xi)}\, e^{t\{-\frac{ac}{a+bd}\xi^2 + 0(\xi^4)\}}$$

$$+ \frac{\sqrt{a+bd}+0(\xi)}{-2(a+bd)+0(\xi)}\, e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

$$+ \frac{\sqrt{a+bd} + 0(\xi)}{2(a+bd) + 0(\xi)}\, e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

$$(1.12)\quad \hat{G}_{13}(t,\xi) = \frac{-b}{(a+bd)+0(\xi)}\, e^{t\{\frac{-ac}{(a+bd)}\xi^2 + 0(\xi^4)\}}$$

$$+ \frac{b}{2(a+bd)+0(\xi)}\, e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

$$+ \frac{b}{2(a+bd)+0(\xi)}\, e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}},$$

$$(1.13)\quad \hat{G}_{22}(t,\xi) = \frac{-abc^2d\xi^2 + 0(\xi^4)}{(a+bd)^3 + 0(\xi)}\, e^{t\{-\frac{ac}{(a+bd)}\xi^2 + 0(\xi^4)\}}$$

$$+ \frac{(a+bd) + 0(\xi)}{2(a+bd) + 0(\xi)}\, e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

$$+ \frac{(a+bd) +0(\xi)}{2(a+bd) + 0(\xi)}\, e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}},$$

$$(1.14)\quad \hat{G}_{23}(t,\xi) = \frac{-iabc\xi + 0(\xi^3)}{(a+bd)^2 + 0(\xi)}\, e^{t\{\frac{-ac}{(a+bd)}\xi^2 + 0(\xi^4)\}}$$

$$+ \frac{b\sqrt{a+bd} + 0(\xi)}{-2(a+bd) + 0(\xi)}\, e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

$$+ \frac{b\sqrt{a+bd} + 0(\xi)}{2(a+bd) + 0(\xi)}\, e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

$$(1.15)\quad \hat{G}_{33}(t,\xi) = \frac{a + 0(\xi^2)}{a+bd + 0(\xi)}\, e^{t\{-\frac{ac}{a+bd}\xi^2 + 0(\xi^4)\}}$$

$$+ \frac{bd + 0(\xi)}{2(a+bd) + 0(\xi)}\, e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

$$+ \frac{bd + 0(\xi)}{2(a+bd) + 0(\xi)} e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2 + 0(\xi^3)\}}$$

and if $|\xi| \geq \eta > \bar{\eta}$,

$$(1.10)^* \quad \hat{G}_{11}(t,\xi) = \frac{0(1)}{\xi^4\{c(c-1) + 0(\frac{1}{\xi^2})\}} e^{t\{-c\xi^2 + 0(1)\}}$$

$$+ \frac{-a\xi^2 + 0(1)}{\xi^4\{1 + 0(\frac{1}{\xi^2})\}} e^{t\{-\xi^2 + 0(1)\}}$$

$$+ \frac{1 + \frac{1}{c}\{bd-a(c+1)\}\frac{1}{\xi^2} + 0(\frac{1}{\xi^4})}{1 + \frac{1}{c}\{bd-a-2ac\}\frac{1}{\xi^2} + 0(\frac{1}{\xi^4})} e^{t\{-a + \frac{abd-a^2c}{c}\frac{1}{\xi^2} + 0(\frac{1}{\xi^4})\}}$$

$$(1.11)^* \quad \hat{G}_{12}(t,\xi) = \frac{i\frac{bd}{c-1} + 0(\frac{1}{\xi^2})}{\xi^3\{c(c-1) + 0(\frac{1}{\xi^2})\}} e^{t\{-c\xi^2 + 0(1)\}}$$

$$+ \frac{i(c-1) + 0(\frac{1}{\xi^2})}{\xi\{(1-c) + 0(\frac{1}{\xi^2})\}} e^{t\{-\xi^2 + 0(1)\}}$$

$$+ \frac{ic + 0(\frac{1}{\xi^2})}{\xi\{c + 0(\frac{1}{\xi^2})\}} e^{t\{-a + 0(\frac{1}{\xi^2})\}}$$

$$(1.12)^* \quad \hat{G}_{13}(t,\xi) = \frac{b}{c(1-c)\xi^2\{1 + 0(\frac{1}{\xi^2})\}} e^{t\{-c\xi^2 + 0(1)\}}$$

$$+ \frac{b}{(c-1)\xi^2\{1 + 0(\frac{1}{\xi^2})\}} e^{t\{-\xi^2 + 0(1)\}}$$

$$+ \frac{-b}{c\xi^2\{1 + 0(\frac{1}{\xi^2})\}} e^{t\{-a + 0(\frac{1}{\xi^2})\}}$$

$$(1.13)^* \qquad \hat{G}_{22}(t,\xi) = \frac{\frac{-cbd}{c-1} + 0(\frac{1}{\xi^2})}{\xi^2\{c(c-1) + 0(\frac{1}{\xi^2})\}} e^{t\{-c\xi^2 + 0(1)\}}$$

$$+ \frac{1 + 0(\frac{1}{\xi^2})}{1 + 0(\frac{1}{\xi^2})} e^{t\{-\xi^2 + 0(1)\}}$$

$$+ \frac{-ac + 0(\frac{1}{\xi^2})}{\xi^2\{c + 0(\frac{1}{\xi^2})\}} e^{t\{-a + 0(\frac{1}{\xi^2})\}}$$

$$(1.14)^* \qquad G_{23}(t,\xi) = \frac{-ibc + 0(\frac{1}{\xi^2})}{\xi\{c(c-1) + 0(\frac{1}{\xi^2})\}} e^{t\{-c\xi^2 + 0(1)\}}$$

$$+ \frac{-ib + 0(\frac{1}{\xi^2})}{\xi\{(1-c) + 0(\frac{1}{\xi^2})\}} e^{t\{-\xi^2 + 0(1)\}}$$

$$+ \frac{-iba + 0(\frac{1}{\xi^2})}{\xi^3\{c + 0(\frac{1}{\xi^2})\}} e^{t\{-a + 0(\frac{1}{\xi^2})\}}$$

$$(1.15)^* \qquad \hat{G}_{33}(t,\xi) = \frac{1 + 0(\frac{1}{\xi^2})}{1 + 0(\frac{1}{\xi^2})} e^{t\{-c\xi^2 + 0(1)\}}$$

$$+ \frac{\frac{bd}{c-1} + 0(\frac{1}{\xi^2})}{\xi^2\{(1-c) + 0(\frac{1}{\xi^2})\}} e^{t\{-\xi^2 + 0(1)\}}$$

$$+ \frac{\frac{abd}{c} + 0(\frac{1}{\xi^2})}{\xi^4\{c + 0(\frac{1}{\xi^2})\}} e^{t\{-a + 0(\frac{1}{\xi^2})\}} ,$$

where ρ is taken so small and η so large that

$$(\frac{bd}{c} + a + \frac{a}{c}) \frac{1}{\eta^2} << 1$$

and the size of each $O(\cdot)$ is only a small fraction of its preceding term.

Proof. Using Lemma 1.2, we can directly compute the residues of $C_{ij}e^{\lambda t}$ to obtain the result.

Now we fix ρ and η such that the statement in Lemma 1.4 holds true. Then we have

Lemma 1.5. The roots of $P(\xi,\lambda) = 0$ belong to a compact subset of $\{\lambda \in \mathbf{C} : \mathrm{Re}(\lambda) < 0\}$ for all $\xi \in R$ with $\rho \leq |\xi| \leq \eta$.

Proof. Suppose this were not true. From the expressions for λ_i's and $\tilde{\lambda}_i$'s in Lemma 1.2, it follows that there should exist $\xi_0 \in [\rho,\eta]$ such that $P(\xi_0,i\mu) = 0$ for $\mu \in R$. But this is impossible, since

$$P(\xi_0,i\mu) = i\{(c\xi_0^4 + a\xi_0^2 + bd\xi_0^2)\mu - \mu^3\} + ac\xi_0^4 - \xi_0^2(c+1)\mu^2$$

cannot be zero for $\xi_0 \in [\rho,\eta]$, $a > 0$, $c > 0$ and $bd > 0$.

From this lemma, it is easily seen that $\hat{G}_{ij}(t,\xi)$ and its derivatives are uniformly bounded analytic functions of (t,ξ) in $(0,\infty) \times (\rho,\eta)$. Furthermore, they decay to zero exponentially fast as time tends to infinity.

Now we begin to analyze each $G_{ij}(t,x)$ in the L^1-setting. Let us define

(1.16) $$\hat{H}_1(t,\xi) = \hat{G}_{11}(t,\xi) - e^{-at},\ H_1(t,x) = F_\xi^{-1}\hat{H}_1(t,\xi) \quad .$$

Lemma 1.6. $H_1(t,x) \in C([0,\infty); L^1)$, $H_1(0,x) = 0$, $\partial_x H_1(t,x) \in C([0,\infty); L^1)$, $\partial_x H_1(0,x) = 0$, $\partial_{xx} H_1(t,x) \in C((0,\infty); M)$ and the following estimates hold:

(1.17) $$\|H_1(t,x)\| \leq M, \quad \text{for all } t \geq 0 \quad ,$$

(1.18) $$\|\partial_x H_1(t,x)\| \leq M(1+t)^{-\frac{1}{2}}, \quad \text{for all } t \geq 0 \quad ,$$

(1.19) $$\|\partial_{xx} H_1(t,x)\| \leq M(t + t^{\frac{1}{6}})^{-1}, \quad \text{for all } t > 0 \quad ,$$

where M is a constant independent of t.

Proof. First we shall obtain estimates for the case $t \geq 1$. Define

$$(1.20)\quad \hat{H}_2(t,\xi) = \hat{H}_1(t,\xi) - \frac{bd}{a+bd}\, e^{-t\frac{ac}{a+bd}\xi^2} - \frac{a}{2(a+bd)}\, e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}}$$

$$- \frac{a}{2(a+bd)}\, e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}} .$$

Then, using Lemmas 1.4, 1.5, we obtain

$$(1.21)\quad \|\hat{H}_2(t,\xi)\| \le Mt^{-1},\ \left\|\frac{\partial}{\partial\xi}\hat{H}_2(t,\xi)\right\|_{L^2} \le Mt^{\frac{1}{4}},\quad \text{for all}\ \ t \ge 1 ,$$

$$(1.22)\quad \|\xi\hat{H}_2(t,\xi)\|_{L^2} \le Mt^{-\frac{5}{4}},\ \left\|\frac{\partial}{\partial\xi}(\xi\hat{H}_2(t,\xi))\right\|_{L^2} \le Mt^{-\frac{1}{4}},\quad \text{for all}\ \ t \ge 1 .$$

By (1.9) with $T = t^{\frac{5}{6}}$, $\beta = 0$ and (1.8) with $T = t$, $\beta = 0$,

$$(1.23)\qquad \|H_2(t,x)\| \le Mt^{-\frac{1}{6}} ,$$

$$(1.24)\qquad \|\partial_x H_2(t,x)\| \le Mt^{-\frac{3}{4}}$$

hold for all $t \ge 1$. By the dominated convergence theorem,

$$\|\hat{H}_2(t_1,\xi) - \hat{H}_2(t_2,\xi)\| \to 0 \quad , \quad \left\|\frac{\partial}{\partial\xi}\hat{H}_2(t_1,\xi) - \frac{\partial}{\partial\xi}\hat{H}_2(t_2,\xi)\right\|_{L^2} \to 0$$

$$\|\xi\hat{H}_2(t_1,\xi) - \xi\hat{H}_2(t_2,\xi)\|_{L^2} \to 0 \quad , \quad \left\|\frac{\partial}{\partial\xi}(\xi\hat{H}_2(t_1,\xi)) - \frac{\partial}{\partial\xi}(\xi\hat{H}_2(t_2,\xi))\right\|_{L^2} \to 0$$

as $t_1 \to t_2$, for $t_1, t_2 \ge 1$. Therefore $H_2(t,x) \in C([1,\infty); L^1)$ and $\partial_x H_2(t,x) \in C([1,\infty); L^1)$. Next we define

$$(1.25)\qquad \hat{H}_3(t,\xi) = \xi^2\hat{H}_2(t,\xi) - ae^{-at} - \frac{abd-a^2c}{c}\, te^{-at} .$$

Then it is easily seen that

$$(1.26)\quad \|\hat{H}_3(t,\xi)\| \le Mt^{-2},\ \left\|\frac{\partial}{\partial\xi}\hat{H}_3(t,\xi)\right\|_{L^2} \le Mt^{-\frac{3}{4}},\quad \text{for all}\ \ t \ge 1 .$$

Hence, by (1.9) with $T = t^{\frac{5}{6}}$, $\beta = 0$, we obtain

$$(1.27) \qquad \|H_3(t,x)\| \le Mt^{-\frac{7}{6}}, \quad \text{for all } t \ge 1 .$$

By the same argument as above, $H_3(t,x) \in C([1,\infty); L^1)$. In the mean time, it is known that

$$(1.28) \qquad F_\xi^{-1} e^{t(i\beta\xi - r\xi^2)} = \frac{1}{2}\frac{1}{\sqrt{\pi r t}} e^{-\frac{(x+\beta t)^2}{4rt}}, \quad \text{for } r > 0 ,$$

$$(1.29) \qquad \left(\frac{\partial}{\partial x}\right)^m \frac{1}{2}\frac{1}{\sqrt{\pi r t}} e^{-\frac{(x+\beta t)^2}{4rt}} \in C((0,\infty); L^1), \quad \text{for all integer } m \ge 0$$

and

$$(1.30) \qquad \left\|\left(\frac{\partial}{\partial x}\right)^m \frac{1}{2}\frac{1}{\sqrt{\pi r t}} e^{-\frac{(x+\beta t)^2}{4rt}}\right\| \le M_{mr} t^{-\frac{m}{2}}, \quad \text{for all integer } m \ge 0,\ t > 0 ,$$

where M_{mr} depends only on m and r. Thus $H_1(t,x) \in C([1,\infty); L^1)$, $\partial_x H_1(t,x) \in C([1,\infty); L^1)$, $\partial_{xx} H_1(t,x) \in C([1,\infty); \mathcal{M})$ and (1.17), (1.18), (1.19) hold for all $t \ge 1$ by taking large M if necessary. Next we analyze $H_1(t,x)$ for $0 \le t \le 1$. From the estimates

$$(1.31) \qquad \|\hat{H}_1(t,\xi)\| \le M, \quad \left\|\frac{\partial}{\partial\xi}\hat{H}_1(t,\xi)\right\| \le M, \quad \text{for } 0 \le t \le 1$$

$$(1.32) \qquad \|\xi\hat{H}_1(t,\xi)\|_{L^2} \le M, \quad \left\|\frac{\partial}{\partial\xi}(\xi\hat{H}_1(t,\xi))\right\|_{L^2} \le M, \quad \text{for } 0 \le t \le 1 ,$$

we obtain

$$(1.33) \qquad \|H_1(t,x)\| \le M, \quad \text{for all } 0 \le t \le 1$$

and

$$(1.34) \qquad \|\partial_x H_1(t,x)\| \le M, \quad \text{for all } 0 \le t \le 1 ,$$

by (1.8), (1.9) with $T = 1$, $\beta = 0$. It is easy to see that $H_1(t,x) \in C([0,1]; L^1)$ and $\partial_x H_1(t,x) \in C([0,1]; L^1)$ by the dominated convergence theorem. Since $\hat{G}(t,\xi)$ is the principal matrix solution of (1.2), $\hat{G}_{11}(0,\xi) = 1$ for each ξ. Hence, $\hat{H}_1(0,\xi) = 0$ for each ξ, from which it follows that $H_1(0,x) = 0$, $\partial_x H_1(0,x) = 0$ in L^1. Finally, we define

$$(1.35) \qquad \hat{H}_4(t,\xi) = \xi^2\hat{H}_1(t,\xi) - \frac{abd-a^2c}{c} te^{-at} - ae^{-at} .$$

Then we find that

(1.36) $$\|\hat{H}_4(t,\xi)\| \le Mt^{-\frac{1}{2}}, \quad \|\frac{\partial}{\partial\xi}\hat{H}_4(t,\xi)\|_{L^2} \le M, \quad \text{for } 0 < t \le 1 ,$$

from which it follows that

(1.37) $$\|H_4(t,x)\| \le Mt^{-\frac{1}{6}}, \quad \text{for all } 0 < t \le 1 ,$$

by (1.9) with $T = t^{\frac{1}{3}}$, $\beta = 0$. $H_4(t,x) \in C((0,1];\, L^1)$ follows from the same argument as before. Therefore, $\partial_{xx}H_1(t,x) \in C((0,1];\, \mathcal{M})$ and (1.19) holds for all $0 < t \le 1$ (with larger M if necessary).

Let us define

(1.38) $$H_5(t,x) = e^{-at}\delta(x) + \partial_x G_{12}(t,x) ,$$

where $\delta(x)$ is the Dirac delta measure. Then, we have

<u>Lemma 1.7</u>. $G_{12}(t,x) \in C([0,\infty);\, L^1)$, $G_{12}(0,x) = 0$, $H_5(t,x) \in C((0,\infty);\, L^1)$, $\partial_x H_5(t,x) \in C((0,\infty);\, L^1)$ and

(1.39) $$\|G_{12}(t,x)\| \le M, \quad \text{for all } t > 0 ,$$

(1.40) $$\|H_5(t,x)\| \le M\left(t^{\frac{1}{2}} + t^{\frac{1}{6}}\right)^{-1}, \quad \text{for all } t > 0 ,$$

(1.41) $$\|\partial_x H_5(t,x)\| \le M(t^{\frac{1}{2}} + t)^{-1}, \quad \text{for all } t > 0 .$$

<u>Proof</u>. First, we define

(1.42) $$\hat{H}_6(t,\xi) = \hat{G}_{12}(t,\xi) - \frac{1}{2\sqrt{a+bd}}\, e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}} + \frac{1}{2\sqrt{a+bd}}\, e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}}$$

and

(1.43) $$\hat{H}_7(t,\xi) = i\xi\hat{H}_6(t,\xi) + e^{-at} .$$

Then, we can easily derive the following estimates:

(1.44) $$\|\hat{H}_6(t,\xi)\|_{L^2} \le Mt^{-\frac{3}{4}}, \quad \|\frac{\partial}{\partial\xi}\hat{H}_6(t,\xi)\|_{L^2} \le Mt^{\frac{1}{4}}, \quad \text{for all } t > 1 ,$$

$$(1.45)\quad \|\hat{H}_7(t,\xi)\| \le Mt^{-\frac{3}{2}},\ \|\frac{\partial}{\partial\xi}\hat{H}_7(t,\xi)\|_{L^2} \le Mt^{-\frac{1}{4}},\ \text{for all } t \ge 1 ,$$

$$(1.46)\quad \|\xi\hat{H}_7(t,\xi)\|_{L^2} \le Mt^{-\frac{7}{4}},\ \|\frac{\partial}{\partial\xi}(\xi\hat{H}_7(t,\xi))\|_{L^2} \le Mt^{-\frac{3}{4}},\ \text{for all } t \ge 1 .$$

With these estimates, we can prove (1.39), (1.40) and (1.41), for $t \ge 1$, analogously to the proof of Lemma 1.6. Next the following estimates

$$(1.47)\quad \|\hat{G}_{12}(t,\xi)\|_{L^2} \le M,\ \|\frac{\partial}{\partial\xi}\hat{G}_{12}(t,\xi)\|_{L^2} \le M,\ \text{for all } 0 \le t \le 1 ,$$

$$(1.48)\quad \|\hat{H}_5(t,\xi)\| \le Mt^{-\frac{1}{2}},\ \|\frac{\partial}{\partial\xi}\hat{H}_5(t,\xi)\|_{L^2} \le M,\ \text{for all } 0 < t \le 1 ,$$

$$(1.49)\quad \|\xi\hat{H}_5(t,\xi)\|_{L^2} \le Mt^{-\frac{3}{4}},\ \|\frac{\partial}{\partial\xi}(\xi\hat{H}_5(t,\xi))\|_{L^2} \le Mt^{-\frac{1}{4}},\ \text{for all } 0 < t \le 1 ,$$

will yield (1.39), for $0 \le t \le 1$, and (1.40), (1.41), for $0 < t \le 1$ (with larger M if necessary). The continuity in t can be proved by the dominated convergence theorem and $G_{12}(0,x) = 0$ in L^1 follows from the property of $\hat{G}(t,\xi)$ as before.

We define

$$(1.50)\qquad H_8(t,x) = \partial_{xx}G_{13}(t,x) - \frac{b}{c}e^{-at}\delta(x) .$$

Then we have

<u>Lemma 1.8</u>. $G_{13}(t,x) \in C([0,\infty); L^1)$, $G_{13}(0,x) = 0$, $\partial_x G_{13}(t,x) \in C([0,\infty); L^1)$, $\partial_x G_{13}(0,x) = 0$, $H_8(t,x) \in C((0,\infty); L^1)$, $\partial_x H_8(t,x) \in C((0,\infty); L^1)$ and

$$(1.51)\qquad \|G_{13}(t,x)\| \le M,\ \text{for all } t \ge 0 ,$$

$$(1.52)\qquad \|\partial_x G_{13}(t,x)\| \le M(1+t)^{-\frac{1}{2}},\ \text{for all } t \ge 0 ,$$

$$(1.53)\qquad \|H_8(t,x)\| \le M(t^{\frac{1}{6}} + t)^{-1},\ \text{for all } t > 0 ,$$

$$(1.54)\qquad \|\partial_x H_8(t,x)\| \le M(t^{\frac{1}{2}} + t^{\frac{3}{2}})^{-1},\ \text{for all } t > 0 .$$

Proof. We start by defining

$$(1.55)\qquad \hat{H}_9(t,\xi) = \hat{G}_{13}(t,\xi) - \frac{b}{2(a+bd)}\, e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}} - \frac{b}{2(a+bd)}\, e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}}$$

and

$$(1.56)\qquad \hat{H}_{10}(t,\xi) = -\xi^2\hat{H}_9(t,\xi) - \frac{b}{c}\, e^{-at} \ .$$

We obtain the following estimates:

$$(1.57)\quad \|\hat{H}_9(t,\xi)\| \le Mt^{-\frac{1}{2}},\ \left\|\frac{\partial}{\partial\xi}\hat{H}_9(t,\xi)\right\|_{L^2} \le Mt^{\frac{1}{4}},\ \text{ for all } t \ge 1 \ ,$$

$$(1.58)\quad \|\xi\hat{H}_9(t,\xi)\|_{L^2} \le Mt^{-\frac{3}{4}},\ \left\|\frac{\partial}{\partial\xi}(\xi\hat{H}_9(t,\xi))\right\|_{L^2} \le Mt^{-\frac{1}{4}},\ \text{ for all } t \ge 1 \ ,$$

$$(1.59)\quad \|\hat{H}_{10}(t,\xi)\| \le Mt^{-\frac{3}{2}},\ \left\|\frac{\partial}{\partial\xi}\hat{H}_{10}(t,\xi)\right\|_{L^2} \le Mt^{-\frac{3}{4}},\ \text{ for all } t \ge 1 \ ,$$

$$(1.60)\quad \|\xi\hat{H}_{10}(t,\xi)\|_{L^2} \le Mt^{-\frac{7}{4}},\ \left\|\frac{\partial}{\partial\xi}(\xi\hat{H}_{10}(t,\xi))\right\|_{L^2} \le Mt^{-\frac{5}{4}},\ \text{ for all } t \ge 1 \ .$$

Combining these inequalities with (1.8), (1.9), we obtain (1.51) to (1.54), for $t \ge 1$. *To consider the case* $t \le 1$, *we list:*

$$(1.61)\quad \|\hat{G}_{13}(t,\xi)\| \le M,\ \left\|\frac{\partial}{\partial\xi}\hat{G}_{13}(t,\xi)\right\|_{L^2} \le M,\ \text{ for all } 0 \le t \le 1 \ ,$$

$$(1.62)\quad \|\xi\hat{G}_{13}(t,\xi)\|_{L^2} \le M,\ \left\|\frac{\partial}{\partial\xi}(\xi\hat{G}_{13}(t,\xi))\right\|_{L^2} \le M,\ \text{ for all } 0 \le t \le 1 \ ,$$

$$(1.63)\quad \|\hat{H}_8(t,\xi)\| \le Mt^{-\frac{1}{2}},\ \left\|\frac{\partial}{\partial\xi}\hat{H}_8(t,\xi)\right\|_{L^2} \le M,\ \text{ for all } 0 < t \le 1 \ ,$$

$$(1.64)\quad \|\xi\hat{H}_8(t,\xi)\|_{L^2} \le Mt^{-\frac{3}{4}},\ \left\|\frac{\partial}{\partial\xi}(\xi\hat{H}_8(t,\xi))\right\|_{L^2} \le Mt^{-\frac{1}{4}},\ \text{ for all } 0 < t \le 1 \ .$$

From these inequalities, we derive (1.51), (1.52), for $0 < t < 1$, and (1.53), (1.54), for $0 < t < 1$. The remaining assertions can be verified by the same method as in the proof of previous lemmas.

We define

$$(1.65) \qquad H_{11}(t,x) = \partial_{xx}G_{22}(t,x) - ae^{-at}\delta(x) \quad ,$$

and state

<u>Lemma 1.9</u>. $G_{22}(t,x) \in C((0,\infty); L^1)$, $\partial_x G_{22}(t,x) \in C((0,\infty); L^1)$, $H_{11}(t,x) \in C((0,\infty); L^1)$ and

$$(1.66) \qquad \|G_{22}(t,x)\| \leq M, \quad \text{for all} \quad t > 0 \quad ,$$

$$(1.67) \qquad \|\partial_x G_{22}(t,x)\| \leq Mt^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad ,$$

$$(1.68) \qquad \|H_{11}(t,x)\| \leq Mt^{-1}, \quad \text{for all} \quad t > 0 \quad .$$

Moreover, for each $f \in L^1(R)$, $G_{22}(t,x)*f(x) \to f(x)$ in L^1 as $t \to 0^+$.

<u>Proof</u>. To consider the case $t > 1$, we define

$$(1.69) \qquad \hat{H}_{12}(t,\xi) = \hat{G}_{22}(t,\xi) - \frac{1}{2}e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}}$$
$$- \frac{1}{2}e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}}$$

and

$$(1.70) \qquad \hat{H}_{13}(t,\xi) = -\xi^2\hat{H}_{12}(t,\xi) - ae^{-at} \quad .$$

Then, we have

$$(1.71) \quad \|\hat{H}_{12}(t,\xi)\| \leq Mt^{-1}, \quad \|\frac{\partial}{\partial\xi}\hat{H}_{12}(t,\xi)\|_{L^2} \leq Mt^{\frac{1}{4}}, \quad \text{for all} \quad t > 1 \quad ,$$

$$(1.72) \quad \|\xi\hat{H}_{12}(t,\xi)\|_{L^2} \leq Mt^{-\frac{5}{4}}, \quad \|\frac{\partial}{\partial\xi}(\xi\hat{H}_{12}(t,\xi))\|_{L^2} \leq Mt^{-\frac{1}{4}}, \quad \text{for all} \quad t > 1 \quad ,$$

$$(1.73) \quad \|\hat{H}_{13}(t,\xi)\| \leq Mt^{-2}, \quad \|\frac{\partial}{\partial\xi}\hat{H}_{13}(t,\xi)\|_{L^2} \leq Mt^{-\frac{3}{4}}, \quad \text{for all} \quad t > 1 \quad .$$

Combining these inequalities with (1.8), (1.9), (1.30), we derive (1.66), (1.67), (1.68) for $t \geq 1$. For the case $t \leq 1$, we define

(1.74) $$\hat{H}_{14}(t,\xi) = \hat{G}_{22}(t,\xi) - e^{-t\xi^2}$$

and

(1.75) $$\hat{H}_{15}(t,\xi) = -\xi^2\hat{H}_{14}(t,\xi) - ae^{-at} \quad .$$

Then the following estimates

(1.76) $$\|\hat{H}_{14}(t,\xi)\| \leq M, \ \|\frac{\partial}{\partial\xi}\hat{H}_{14}(t,\xi)\|_{L^2} \leq M, \quad \text{for all} \quad 0 \leq t \leq 1 \quad ,$$

(1.77) $$\|\xi\hat{H}_{14}(t,\xi)\|_{L^2} \leq M, \ \|\frac{\partial}{\partial\xi}(\xi\hat{H}_{14}(t,\xi))\|_{L^2} \leq M, \quad \text{for all} \quad 0 \leq t \leq 1 \quad ,$$

(1.78) $$\|\hat{H}_{15}(t,\xi)\| \leq Mt^{-\frac{1}{2}}, \ \|\frac{\partial}{\partial\xi}\hat{H}_{15}(t,\xi)\|_{L^2} \leq M, \quad \text{for all} \quad 0 < t \leq 1 \quad ,$$

are combined with (1.8), (1.9), (1.30) to yield (1.66), (1.67), (1.68) for $0 < t \leq 1$. In particular, $H_{14}(t,x) \to 0$ in $L^1(R)$ as $t \to 0$, from which the last assertion of the lemma follows.

<u>Lemma 1.10</u>. $G_{23}(t,x) \in C([0,\infty); L^1)$, $G_{23}(0,x) = 0$, $\partial_x G_{23}(t,x) \in C((0,\infty); L^1)$, $\partial_{xx}G_{23}(t,x) \in C((0,\infty); L^1)$, $\partial_{xxx}G_{23}(t,x) \in C((0,\infty); \mathcal{M})$ and

(1.79) $$\|G_{23}(t,x)\| \leq M, \quad \text{for all} \quad t \geq 0 \quad ,$$

(1.80) $$\|\partial_x G_{23}(t,x)\| \leq M(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad ,$$

(1.81) $$\|\partial_{xx}G_{23}(t,x)\| \leq M(t^{\frac{1}{2}} + t)^{-1}, \quad \text{for all} \quad t > 0 \quad ,$$

(1.82) $$\|\partial_{xxx}G_{23}(t,x)\| \leq M(t^{\frac{3}{2}} + t)^{-1}, \quad \text{for all} \quad t > 0 \quad .$$

<u>Proof</u>. We define

$$(1.83)\qquad \hat{H}_{16}(t,\xi) = \hat{G}_{23}(t,\xi) - \frac{b}{2\sqrt{a+bd}}\, e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}}$$

$$+ \frac{b}{2\sqrt{a+bd}}\, e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}}, \quad \text{for all} \quad t > 1 ,$$

$$(1.84)\qquad \hat{H}_{17}(t,\xi) = -i\xi^3\hat{H}_{16}(t,\xi) + \frac{ba}{c}\, e^{-at}, \quad \text{for} \quad t > 1 ,$$

$$(1.85)\qquad \hat{H}_{18}(t,\xi) = i\xi G_{23} - \frac{b}{1-c}\, e^{-t\xi^2} + \frac{b}{1-c}\, e^{-tc\xi^2}, \quad \text{for} \quad 0 < t < 1 ,$$

$$(1.86)\qquad \hat{H}_{19}(t,\xi) = -\xi^2\hat{H}_{18}(t,\xi) + \frac{ba}{c}\, e^{-at}, \quad \text{for} \quad 0 < t < 1 .$$

Then, proceeding as in previous lemmas, we can derive (1.79) to (1.82) from the following inequalities:

$$(1.87)\qquad \|\hat{H}_{16}(t,\xi)\| < Mt^{-1}, \ \left\|\frac{\partial}{\partial\xi}\hat{H}_{16}(t,\xi)\right\|_{L^2} < Mt^{\frac{1}{4}}, \quad \text{for all} \quad t > 1 ,$$

$$(1.88)\qquad \|\xi\hat{H}_{16}(t,\xi)\| < Mt^{-\frac{3}{2}}, \ \left\|\frac{\partial}{\partial\xi}(\xi\hat{H}_{16}(t,\xi))\right\|_{L^2} < Mt^{-\frac{1}{4}}, \quad \text{for all} \quad t > 1 ,$$

$$(1.89)\qquad \|\xi^2\hat{H}_{16}(t,\xi)\|_{L^2} < Mt^{-\frac{7}{4}}, \ \left\|\frac{\partial}{\partial\xi}(\xi^2\hat{H}_{16}(t,\xi))\right\|_{L^2} < Mt^{-\frac{3}{4}}, \quad \text{for all} \quad t > 1 ,$$

$$(1.90)\qquad \|\hat{H}_{17}(t,\xi)\| < Mt^{-\frac{5}{2}}, \ \left\|\frac{\partial}{\partial\xi}\hat{H}_{17}(t,\xi)\right\|_{L^2} < Mt^{-\frac{5}{4}}, \quad \text{for all} \quad t > 1 ,$$

$$(1.91)\qquad \|\hat{G}_{23}(t,\xi)\|_{L^2} < M, \ \left\|\frac{\partial}{\partial\xi}\hat{G}_{23}(t,\xi)\right\|_{L^2} < M, \quad \text{for all} \quad 0 < t < 1 ,$$

$$(1.92)\qquad \|\hat{H}_{18}(t,\xi)\| < M, \ \left\|\frac{\partial}{\partial\xi}\hat{H}_{18}(t,\xi)\right\|_{L^2} < M, \quad \text{for all} \quad 0 < t < 1 ,$$

$$(1.93)\qquad \|\xi\hat{H}_{18}(t,\xi)\|_{L^2} < M, \ \left\|\frac{\partial}{\partial\xi}(\xi\hat{H}_{18}(t,\xi))\right\|_{L^2} < M, \quad \text{for all} \quad 0 < t < 1 ,$$

$$(1.94)\qquad \|\hat{H}_{19}(t,\xi)\| < Mt^{-\frac{1}{2}}, \ \left\|\frac{\partial}{\partial\xi}\hat{H}_{19}(t,\xi)\right\|_{L^2} < M, \quad \text{for all} \quad 0 < t < 1 .$$

__Lemma 1.11__. $(\frac{\partial}{\partial x})^m G_{33}(t,x) \in C((0,\infty); L^1)$, $m = 0,1,2,3$, and

$$\|(\frac{\partial}{\partial x})^m G_{33}(t,x)\| \le Mt^{-\frac{m}{2}}, \quad \text{for all } t > 0, \ m = 0,1,2,3 \quad , \tag{1.95}$$

$$\int_{-\infty}^{\infty} \partial_{xx} G_{33}(t,x)dx = \int_{-\infty}^{\infty} \partial_{xxx} G_{33}(t,x)dx = 0, \quad \text{for all } t > 0 \quad . \tag{1.96}$$

Moreover, for each $f \in L^1(R)$, $G_{33}(t,x)*f(x) \to f(x)$ in L^1 as $t \to 0^+$, and if $0 < \lambda < \frac{1}{2}$, it holds that $|x|^\lambda \partial_{xx} G_{33}(t,x)$, $|x|^\lambda \partial_{xxx} G_{33}(t,x) \in C((0,\infty); L^1)$ with

$$\| \ |x|^\lambda \partial_{xx} G_{33}(t,x)\| \le M(t^{-1+\frac{\lambda}{2}} + t^{-1+\lambda}) \tag{1.97}$$

and

$$\| \ |x|^\lambda \partial_{xxx} G_{33}(t,x)\| \le M\left(t^{-\frac{3}{2}+\frac{\lambda}{2}} + t^{-\frac{3}{2}+\lambda}\right) \tag{1.98}$$

for all $t > 0$.

__Proof__. We define

$$\hat{H}_{20}(t,\xi) = \hat{G}_{33}(t,\xi) - \frac{bd}{2(a+bd)} e^{t\{i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}} - \frac{bd}{2(a+bd)} e^{t\{-i\sqrt{a+bd}\,\xi - \frac{a+bd(c+1)}{2(a+bd)}\xi^2\}} \quad , \tag{1.99}$$

$$\hat{H}_{21}(t,\xi) = \hat{G}_{33}(t,\xi) - e^{-tc\xi^2} \quad . \tag{1.100}$$

Then, we obtain the estimates:

$$\|\hat{H}_{20}(t,\xi)\| \le Mt^{-\frac{1}{2}}, \ \|\frac{\partial}{\partial\xi}\hat{H}_{20}(t,\xi)\|_{L^2} \le Mt^{\frac{1}{4}}, \quad \text{for all } t \ge 1 \quad , \tag{1.101}$$

$$\|\xi\hat{H}_{20}(t,\xi)\| \le Mt^{-1}, \ \|\frac{\partial}{\partial\xi}(\xi\hat{H}_{20}(t,\xi))\|_{L^2} \le Mt^{-\frac{1}{4}}, \quad \text{for all } t \ge 1 \quad , \tag{1.102}$$

$$\|\xi^2\hat{H}_{20}(t,\xi)\| \le Mt^{-\frac{3}{2}}, \ \|\frac{\partial}{\partial\xi}(\xi^2\hat{H}_{20}(t,\xi))\|_{L^2} \le Mt^{-\frac{3}{4}}, \quad \text{for all } t \ge 1 \quad , \tag{1.103}$$

(1.104) $\|\xi^3\hat{H}_{20}(t,\xi)\|_{L^2} \leq Mt^{-\frac{7}{4}}$, $\|\frac{\partial}{\partial\xi}(\xi^3\hat{H}_{20}(t,\xi))\|_{L^2} \leq Mt^{-\frac{5}{4}}$, for all $t \geq 1$,

(1.105) $\|\hat{H}_{21}(t,\xi)\| \leq M$, $\|\frac{\partial}{\partial\xi}\hat{H}_{21}(t,\xi)\|_{L^2} \leq M$, for all $0 \leq t \leq 1$,

(1.106) $\|\xi\hat{G}_{33}(t,\xi)\|_{L^2} \leq Mt^{-\frac{3}{4}}$, $\|\frac{\partial}{\partial\xi}(\xi\hat{G}_{33}(t,\xi))\|_{L^2} \leq Mt^{-\frac{1}{4}}$, for all $0 < t \leq 1$,

(1.107) $\|\xi^2\hat{G}_{33}(t,\xi)\| \leq Mt^{-\frac{3}{2}}$, $\|\frac{\partial}{\partial\xi}(\xi^2\hat{G}_{33}(t,\xi))\|_{L^2} \leq Mt^{-\frac{3}{4}}$, for all $0 < t \leq 1$,

(1.108) $\|\xi^3\hat{G}_{33}(t,\xi)\|_{L^2} \leq Mt^{-\frac{7}{4}}$, $\|\frac{\partial}{\partial\xi}(\xi^3\hat{G}_{33}(t,\xi))\|_{L^2} \leq Mt^{-\frac{5}{4}}$, for all $0 < t \leq 1$.

Using these inequalities and (1.8), (1.9) with suitable $T > 0$, we arrive at (1.95). Combining (1.8), (1.9) with

(1.109) $$\| \, |x|^\lambda \partial_{xx} \frac{1}{\sqrt{\pi rt}} e^{-\frac{(x+\beta t)^2}{4rt}} \| \leq M_{\beta r\lambda}(t^{-1+\frac{\lambda}{2}} + t^{-1+\lambda}), \quad \text{for all } t > 0 ,$$

(1.110) $$\| \, |x|^\lambda \partial_{xxx} \frac{1}{\sqrt{\pi rt}} e^{-\frac{(x+\beta t)^2}{4rt}} \| \leq \tilde{M}_{\beta r\lambda}(t^{-\frac{3}{2}+\frac{\lambda}{2}} + t^{-\frac{3}{2}+\lambda}), \text{ for all } t > 0 ,$$

where $r > 0$, $0 < \lambda < 1$, $M_{\beta r\lambda}$ and $\tilde{M}_{\beta r\lambda}$ depend only on β, r, λ, we get (1.97) and (1.98). The continuity in t can be verified in the same way as before and (1.96) is an immediate consequence of the first statement of the lemma.

With the aid of Lemmas 1.6 to 1.11, we can discuss the properties of solutions to (1.1), (0.7). First of all, we need to observe:

Lemma 1.12. If $(u_0,v_0,\theta_0) \in [L^1(R)]^3$, then there is a solution to (1.1), (0.7) in the form

$$
(1.111) \qquad \begin{pmatrix} u(t,x) \\ v(t,x) \\ \theta(t,x) \end{pmatrix} = G(t,x) * \begin{pmatrix} u_0(x) \\ v_0(x) \\ \theta_0(x) \end{pmatrix} ,
$$

which is the unique solution within the function class of $[C([0,T];\, L^1)]^3$ for any $T > 0$.

<u>Proof</u>. On account of the properties of $G(t,x)$ stated in Lemmas 1.6 to 1.11, the right-hand side of (1.111) belongs to $[C([0,\infty);\, L^1)]^3$ and satisfies (0.7). By taking the Fourier transform of (1.111), it is easily seen that (1.111) is a solution to (1.1) in the sense of distribution. The uniqueness can be verified by the standard argument which proceeds as follows: suppose $(U(t,x),\, V(t,x),\, \Theta(t,x)) \in [C([0,T];\, L^1)]^3$ is a solution of (1.1) with the zero initial condition. Since the Fourier transformation is a continuous mapping from $L^1(R)$ to $C_0(R)$, $(\hat{U}(t,\xi),\, \hat{V}(t,\xi),\, \hat{\Theta}(t,\xi)) \in [C([0,T];\, C_0)]^3$ and satisfies (1.2) in $\mathcal{D}^*((0,T) \times R)$. Hence, for each $\varepsilon > 0$ and each $\zeta \in R$, it holds that

$$
(1.112) \quad -\int_{-\infty}^{\infty} \int_0^T \begin{pmatrix} \hat{U}(t,\xi) \\ \hat{V}(t,\xi) \\ \hat{\Theta}(t,\xi) \end{pmatrix} \partial_t \phi(t) \rho_\varepsilon(\zeta-\xi)\, dt\, d\xi = \int_{-\infty}^{\infty} \int_0^T \hat{A}(\xi) \begin{pmatrix} \hat{U}(t,\xi) \\ \hat{V}(t,\xi) \\ \hat{\Theta}(t,\xi) \end{pmatrix} \phi(t) \rho_\varepsilon(\zeta-\xi)\, dt\, d\xi ,
$$

for all $\phi \in C_0^\infty((0,T))$, from which it follows that, by passing to the limit,

$$
(1.113) \qquad \frac{\partial}{\partial t} \begin{pmatrix} \hat{U}(t,\zeta) \\ \hat{V}(t,\zeta) \\ \hat{\Theta}(t,\zeta) \end{pmatrix} = \hat{A}(\zeta) \begin{pmatrix} \hat{U}(t,\zeta) \\ \hat{V}(t,\zeta) \\ \hat{\Theta}(t,\zeta) \end{pmatrix}
$$

holds for each fixed $\zeta \in R$ in $\mathcal{D}^*((0,T))$, hence in the classical sense. Therefore, $\hat{U}(t,\zeta) = \hat{V}(t,\zeta) = \hat{\Theta}(t,\zeta) \equiv 0$ for all $t \in [0,T]$ and $\zeta \in R$.

Now we state the regularity and the asymptotic behavior of solutions to (1.1), (0.7):

<u>Theorem 1.13</u>. Let $(u_0,v_0,\theta_0) \in [L^1 \cap BV]^3$ and $(u(t,x),v(t,x),\theta(t,x))$ be the unique solution to (1.1), (0.7) in Lemma 1.12. Let $\|u_0\| + \|u_{0x}\| + \|v_0\|$ $\|v_{0x}\| + \|\theta_0\| + \|\theta_{0x}\| = \mu > 0$, and fix any integer $m \geq 2$ and any real number $0 < \alpha \leq \frac{1}{3}$. Then, we have:

(i) $u(t,x) = w(t,x) + z(t,x)$, where $w(t,x) \in ([0,\infty); L^1)$, $w(0,x) = u_0(x)$, $\partial_x w(t,x) \in C([0,\infty); M)$, $\partial_t w(t,x) \in C((0,\infty); L^1)$, $\partial_t \partial_x w(t,x) \in C((0,\infty); M)$, $z(t,x) \in C([0,\infty); L^1)$, $z(0,x) = 0$, $\partial_x z(t,x) \in C([0,\infty); L^1)$, $\partial_{xx} z(t,x) \in C((0,\infty); M)$ and

$$\|w(t,x)\| \leq \mu M(1+t)^{\frac{1-m}{2}}, \quad \text{for all } t \geq 0, \tag{1.114}$$

$$\|\partial_x w(t,x)\| \leq \mu M(1+t)^{-\frac{m}{2}}, \quad \text{for all } t \geq 0, \tag{1.115}$$

$$\|\partial_t w(t,x)\| \leq \mu M(1+t)^{-\frac{m}{2}}, \quad \text{for all } t > 0, \tag{1.116}$$

$$\|\partial_t \partial_x w(t,x)\| \leq \mu M(t^{-\frac{1}{2}} + t^{-\frac{\alpha}{2}})(1+t)^{-\frac{m}{2}}, \quad \text{for all } t > 0, \tag{1.117}$$

$$\|z(t,x)\| \leq \mu M, \quad \text{for all } t \geq 0, \tag{1.118}$$

$$\|\partial_x z(t,x)\| \leq \mu M(1+t)^{-\frac{1}{2}}, \quad \text{for all } t \geq 0, \tag{1.119}$$

$$\|\partial_{xx} z(t,x)\| \leq \mu M t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0. \tag{1.120}$$

(ii) $v(t,x) \in C([0,\infty); L^1)$, $v(0,x) = v_0(x)$, $\partial_x v(t,x) \in C((0,\infty); L^1)$,

$\partial_t v(t,x) \in C((0,\infty); M)$, $\partial_{xx} v(t,x) \in C((0,\infty); M)$ and

(1.121) $$\|v(t,x)\| \le \mu M, \quad \text{for all } t > 0 ,$$

(1.122) $$\|\partial_x v(t,x)\| \le \mu M(1+t)^{-\frac{1}{2}}, \quad \text{for all } t > 0 ,$$

(1.123) $$\|\partial_{xx} v(t,x)\| \le \mu M t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 ,$$

(1.124) $$\|\partial_t v(t,x)\| \le \mu M t^{-\frac{1}{2}}, \quad \text{for all } t > 0 .$$

(iii) $\partial_t u(t,x) = \partial_x v(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$

(iv) $\theta(t,x) \in C([0,\infty); L^1)$, $\theta(0,x) = \theta_0(x)$, $\partial_x \theta(t,x) \in C((0,\infty); L^1)$,

$\partial_t \theta(t,x) \in C((0,\infty); L^1)$, $\partial_{xx}\theta(t,x) \in C((0,\infty); \Lambda_\alpha^{1,\infty})$ and

(1.125) $$\|\theta(t,x)\| \le \mu M, \quad \text{for all } t > 0 ,$$

(1.126) $$\|\partial_x \theta(t,x)\| \le \mu M(1+t)^{-\frac{1}{2}}, \quad \text{for all } t > 0 ,$$

(1.127) $$\|\partial_{xx}\theta(t,x)\| \le \mu M t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 ,$$

(1.128) $$\|\partial_t \theta(t,x)\| \le \mu M t^{-\frac{1}{2}}, \quad \text{for all } t > 0 ,$$

(1.129) $$|||\partial_{xx}\theta(t,x)|||_\alpha \le \mu M t^{\frac{-1-\alpha}{2}} (1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 ,$$

(1.130) $$\|\partial_t \theta(t,x) - d\partial_x v(t,x)\| \le \mu M t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 .$$

All the above M's are constants independent of μ and t.

Proof. By defining

$$w(t,x) = e^{-at}u_0(x),$$

$$z(t,x) = H_1(t,x)*u_0(x) + G_{12}(t,x)*v_0(x) + G_{13}(t,x)*\theta_0(x) \quad ,$$

we can easily verify the properties (i) with the aid of Lemmas 1.6 to 1.8. Also, by virture of (1.1), (1.111) and Lemmas 1.7 to 1.11, it is easy to derive all the other properties except (1.129) and the continuity of $\partial_{xx}\theta(t,x)$ in $\Lambda_\alpha^{1,\infty}$. Similarly we can prove

$$(1.131) \qquad \begin{cases} \partial_{xxx}\theta(t,x) \in C((0,\infty);\ M) \\ \|\partial_{xxx}\theta(t,x)\| \le \mu M\left(t+t^{\frac{3}{2}}\right)^{-1}, \quad \text{for all} \quad t > 0 \quad , \end{cases}$$

and a sharp version of (1.127):

$$(1.132) \qquad \|\partial_{xx}\theta(t,x)\| \le \mu M\left(t^{\frac{1}{2}} + t\right)^{-1}, \quad \text{for all} \quad t > 0 \quad .$$

Now the proof is completed by combining (1.131), (1.132) with the following lemma.

<u>Lemma 1.14</u>. Suppose $f(t,x) \in C((0,\infty);\ L^1 \cap BV)$ satisfying

$$\|f(t,x)\| \le \left(t^{\frac{1}{2}} + t\right)^{-1} \quad \text{and} \quad \|\partial_x f(t,x)\| \le \left(t + t^{\frac{3}{2}}\right)^{-1}, \quad \text{for all} \quad t > 0 \quad .$$

Then, $f(t,x) \in C((0,\infty);\ \Lambda_\beta^{1,\infty})$ and

$$(1.133) \qquad \begin{aligned} |||f(t,x)|||_\beta &\le Mt^{-\frac{\beta}{2}}\left(t^{\frac{1}{2}} + t\right)^{-1} \\ &\le Mt^{\frac{-1-\beta}{2}}(1+t)^{-\frac{\beta}{2}} \end{aligned}$$

holds for all $t > 0$, where $0 < \beta < 1$ and the constants M are independent of t.

<u>Proof</u>. We need the following fact: for each $\phi \in L^1 \cap BV$,

$$\frac{1}{|h|^\beta}\|\phi(x+h) - \phi(x)\| \le |h|^{1-\beta}\|\partial_x\phi\|$$

holds for any $h \neq 0$. Indeed, if $\phi \in L^1 \cap BV$, there is a sequence $\{\phi_n\}_{n=1}^{\infty}$ such that $\phi_n \in C^{\infty}$, $\phi_n \to \phi$ in L^1 and $\|\partial_x \phi_n\| \leq \|\partial_x \phi\|$, for all $n \geq 1$, from which it follows that

$$\|\phi(x+h) - \phi(x)\| = \lim_{n\to\infty} \|\phi_n(x+h) - \phi_n(x)\| = \left\|\int_0^h \partial_x \phi_n(x+\zeta)d\zeta\right\|$$

$$\leq \varliminf_{n\to\infty} |h| \, \|\partial_x \phi_n\| \leq |h| \, \|\partial_x \phi\| \quad .$$

Now, if $0 < |h| \leq \sqrt{t}$,

$$(1.134) \qquad \frac{1}{|h|^{\beta}} \|f(t,x+h) - f(t,x)\| \leq |h|^{1-\beta} \|\partial_x f(t,x)\| \leq t^{\frac{1-\beta}{2}} \|\partial_x f(t,x)\|$$

$$\leq t^{\frac{1-\beta}{2}} \left(t + t^{\frac{3}{2}}\right)^{-1} \quad ,$$

and if $0 < \sqrt{t} \leq |h|$,

$$(1.135) \qquad \frac{1}{|h|^{\beta}} \|f(t,x+h) - f(t,x)\| \leq 2t^{-\frac{\beta}{2}} \|f(t,x)\| \leq 2t^{-\frac{\beta}{2}}\left(t^{\frac{1}{2}} + t\right)^{-1} \quad .$$

Considering the case $0 < t \leq 1$ and the case $1 \leq t$, separately, (1.133) is easily obtained from (1.134), (1.135). Next, we observe that (1.134), (1.135) also imply that

$$(1.136) \qquad |||\phi(x)|||_{\beta} \leq \|\partial_x \phi\| + 2\|\phi\|$$

holds for all $\phi \in L^1 \cap BV$, from which we deduce that $f(t,x) \in C((0,\infty); \Lambda_{\beta}^{1,\infty})$.

Remark 1.15. In fact, some of the estimates stated in Theorem 1.13 are not sharp (e.g., compare (1.127) and (1.132)). They are, however, in such weak form as to be applied directly to the nonlinear problem.

2. Nonlinear Problem

In this section we will establish our main result:

Theorem 2.1. Assume (0.9) and (0.10). Then, there exists a positive number δ such that if $(u_0(x), v_0(x), \theta_0(x)) \in (L^1 \cap BV)^3$ and $\|u_0\| + \|\partial_x u_0\| + \|v_0\| + \|\partial_x v_0\| + \|\theta_0\| + \|\partial_x \theta_0\| < \delta$, there is a global solution $(u(t,x), v(t,x), \theta(t,x))$ to (0.6), (0.7), satisfying the properties (i) to (iv) (with different constants if necessary) stated in Theorem 1.13.

The proof of this theorem will be split into three steps. First, we construct a suitable function space χ with the properties which were found for the linear problem. Second, we define a mapping T from χ into itself so that the fixed point of T may be a solution of (0.11). Finally, we prove that the mapping T is a contraction and that the solution to (0.11), (0.7) is also the solution to (0.6), (0.7).

(Step I). We construct χ as follows: Let χ be the set of all quadruplet $(w(t,x), z(t,x), v(t,x), \theta(t,x))$ satisfying the properties (A) to (E):

(A) $w(t,x) \in C([0,\infty); L^1)$, $w(0,x) = u_0(x)$, $\partial_x w(t,x) \in C([0,\infty); M)$, $\partial_t w(t,x) \in C((0,\infty); L^1)$, $\partial_t \partial_x w(t,x) \in C((0,\infty); M)$ with

$$\|w(t,x)\| \leq K(1+t)^{\frac{1-m}{2}}, \quad \text{for all } t \geq 0, \tag{2.1}$$

$$\|\partial_x w(t,x)\| \leq K(1+t)^{-\frac{m}{2}}, \quad \text{for all } t \geq 0, \tag{2.2}$$

$$\|\partial_t w(t,x)\| \leq K(1+t)^{-\frac{m}{2}}, \quad \text{for all } t > 0, \tag{2.3}$$

$$\|\partial_t \partial_x w(t,x)\| \leq K(t^{-\frac{1}{2}} + t^{-\frac{\alpha}{2}})(1+t)^{-\frac{m}{2}}, \quad \text{for all } t > 0, \tag{2.4}$$

where m, α are the numbers fixed in Theorem 1.13, K is a constant independent of t and will be determined after we can collect all the conditions on K.

(B) $z(t,x) \in C([0,\infty); L^1)$, $z(0,x) = 0$, $\partial_x z(t,x) \in C([0,\infty); L^1)$

$\partial_{xx} z(t,x) \in C((0,\infty); \mathcal{M})$ with

$$\|z(t,x)\| \leq K, \quad \text{for all} \quad t \geq 0 \quad , \tag{2.5}$$

$$\|\partial_x z(t,x)\| \leq K(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t \geq 0 \quad , \tag{2.6}$$

$$\|\partial_{xx} z(t,x)\| \leq Kt^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad , \tag{2.7}$$

(C) $\partial_t w(t,x) + \partial_t z(t,x) = \partial_x v(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$.

(D) $v(t,x) \in C([0,\infty); L^1)$, $v(0,x) = v_0(x)$, $\partial_x v(t,x) \in C((0,\infty); L^1)$,

$\partial_t v(t,x) \in C((0,\infty); \mathcal{M})$, $\partial_{xx} v(t,x) \in C((0,\infty); \mathcal{M})$ with

$$\|v(t,x)\| \leq K, \quad \text{for all} \quad t \geq 0 \quad , \tag{2.8}$$

$$\|\partial_x v(t,x)\| \leq K(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad , \tag{2.9}$$

$$\|\partial_{xx} v(t,x)\| \leq Kt^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 \quad , \tag{2.10}$$

$$\|\partial_t v(t,x)\| \leq Kt^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad . \tag{2.11}$$

(E) $\theta(t,x) \in C([0,\infty); L^1)$, $\theta(0,x) = \theta_0(x)$, $\partial_x \theta(t,x) \in C((0,\infty); L^1)$,

$\partial_t \theta(t,x) \in C((0,\infty); L^1)$, $\partial_{xx} \theta(t,x) \in C((0,\infty); \Lambda_\alpha^{1,\infty})$ with

$$\|\theta(t,x)\| \leq K, \quad \text{for all} \quad t \geq 0 \quad , \tag{2.12}$$

$$\|\partial_x \theta(t,x)\| \leq K(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad , \tag{2.13}$$

$$\|\partial_{xx} \theta(t,x)\| \leq Kt^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad , \tag{2.14}$$

$$(2.15)\qquad |||\partial_{xx}\theta(t,x)|||_\alpha \le Kt^{\frac{-1-\alpha}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 ,$$

$$(2.16)\qquad \|\partial_t\theta(t,x)\| \le Kt^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 ,$$

$$(2.17)\qquad \|\partial_t\theta(t,x) - d\partial_x v(t,x)\| \le Kt^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 .$$

Since the solution to (1.1), (0.7) satisfies the properties (A) to (E) if $\mu M \le K$ (see Theorem 1.13), the set χ is not empty. Now χ shall be endowed with the metric $d(\cdot,\cdot)$: for (w,z,v,θ), $(\tilde{w},\tilde{z},\tilde{v},\tilde{\theta}) \in \chi$, we define

$$
\begin{aligned}
(2.18)\ d((w,z,v,\theta),\ (\tilde{w},\tilde{z},\tilde{v},\tilde{\theta})) &= \sup_{t>0}\,(1+t)^{\frac{m-1}{2}}\|w(t,x) - \tilde{w}(t,x)\| \\
&+ \sup_{t>0}\,(1+t)^{\frac{m}{2}}\|\partial_x w(t,x) - \partial_x\tilde{w}(t,x)\| + \sup_{t>0}\,(1+t)^{\frac{m}{2}}\|\partial_t w(t,x) - \partial_t\tilde{w}(t,x)\| \\
&+ \sup_{0<t\le 1}\, t^{\frac{1}{2}}(1+t)^{\frac{m}{2}}\|\partial_t\partial_x w(t,x) - \partial_t\partial_x\tilde{w}(t,x)\| \\
&+ \sup_{t>1}\, t^{\frac{\alpha}{2}}(1+t)^{\frac{m}{2}}\|\partial_t\partial_x w(t,x) - \partial_t\partial_x\tilde{w}(t,x)\| \\
&+ \sup_{t>0}\,\|z(t,x) - \tilde{z}(t,x)\| + \sup_{t>0}\,(1+t)^{\frac{1}{2}}\|\partial_x z(t,x) - \partial_x\tilde{z}(t,x)\| \\
&+ \sup_{t>0}\,(1+t)^{\frac{1}{2}}\|\partial_t z(t,x) - \partial_t\tilde{z}(t,x)\| \\
&+ \sup_{t>0}\, t^{\frac{1}{2}}(1+t)^{\frac{\alpha}{2}}\|\partial_{xx} z(t,x) - \partial_{xx}\tilde{z}(t,x)\| + \sup_{t>0}\,\|v(t,x) - \tilde{v}(t,x)\| \\
&+ \sup_{t>0}\,(1+t)^{\frac{1}{2}}\|\partial_x v(t,x) - \partial_x\tilde{v}(t,x)\| + \sup_{t>0}\, t^{\frac{1}{2}}(1+t)^{\frac{\alpha}{2}}\|\partial_{xx} v(t,x) - \partial_{xx}\tilde{v}(t,x)\| \\
&+ \sup_{t>0}\, t^{\frac{1}{2}}\|\partial_t v(t,x) - \partial_t\tilde{v}(t,x)\| + \sup_{t>0}\,\|\theta(t,x) - \tilde{\theta}(t,x)\|
\end{aligned}
$$

$$+\sup_{t>0}(1+t)^{\frac{1}{2}}\|\partial_x\theta(t,x)-\partial_x\tilde{\theta}(t,x)\|+\sup_{t>0}t^{\frac{1}{2}}(1+t)^{\frac{\alpha}{2}}\|\partial_{xx}\theta(t,x)-\partial_{xx}\tilde{\theta}(t,x)\|$$

$$+\sup_{t>0}t^{\frac{1}{2}}\|\partial_t\theta(t,x)-\partial_t\tilde{\theta}(t,x)\|+\sup_{t>0}t^{\frac{1+\alpha}{2}}(1+t)^{\frac{\alpha}{2}}|||\partial_{xx}\theta(t,x)-\partial_{xx}\tilde{\theta}(t,x)|||_\alpha$$

$$+\sup_{t>0}t^{\frac{1}{2}}(1+t)^{\frac{\alpha}{2}}\|\partial_t\theta(t,x)-d\partial_x v(t,x)-\partial_t\tilde{\theta}(t,x)+d\partial_x\tilde{v}(t,x)\| \quad .$$

It is not difficult to see that χ becomes a complete metric space with the metric $d(\cdot,\cdot)$. The proof of this fact is left to the reader.

Before proceeding to Step (II), we shall make some preliminary remarks. We recall that $p(u,\theta)$ and $\frac{1}{e_\theta(u,\theta)}$ are analytic functions of u, θ in a neighborhood of $(0,0)$. So the first condition we should impose on K is

$$K \le \min(\tfrac{1}{3}\nu,\ 1) \quad , \tag{2.19}$$

where ν is a positive number such that $p(u,\theta)$, $\frac{1}{e_\theta(u,\theta)}$ can be expanded as Taylor series in u, θ if $|u| \le 2\nu$, $|\theta| \le 2\nu$. Hence, recalling that $-p_u(0,0) = a$, we see that

$$p_u(w+z,\ \theta) + a = \sum_{1\le q+r+s}^{\infty} a_{qrs}w^q z^r \theta^s \tag{2.20}$$

is valid if $|w|$, $|z|$, $|\theta| \le 2K$. Next we observe that if $(w,z,v,\theta) \in \chi$, it follows that z, $\theta \in C((0,\infty);\ C_0)$. Hence, for nonnegative integers q, r, s, $(w^{q+1})_x z^r\theta^s$ is well-defined and belongs to $C((0,\infty);\ M)$. Now we define for given $(w,z,v,\theta) \in \chi$,

$$S_n(t,x) = \sum_{1\le q+r+s}^{n} \frac{a_{qrs}}{q+1}(w^{q+1})_x z^r\theta^s \tag{2.21}$$

and

$$\sigma(t,x) = p(w+z,\ \theta)_x - p_u(w+z,\ \theta)z_x - p_\theta(w+z,\ \theta)\theta_x + aw_x \quad . \tag{2.22}$$

Then we have

Lemma 2.2. $S_n(t,x)$, $\sigma(t,x) \in C((0,\infty);\,\mathcal{M})$ and $S_n(t,x) \to \sigma(t,x)$ in $\mathcal{M}$ uniformly in t as $n \to \infty$. In addition, it holds that

$$(2.23) \qquad \|\sigma(t,x)\| \le MK^2(1+t)^{\frac{-1-m}{2}}, \quad \text{for all } t > 0 ,$$

where M is independent of K and t.

Proof. Let us set $w_\varepsilon = w*\rho_\varepsilon$, $z_\varepsilon = z*\rho_\varepsilon$, $\theta_\varepsilon = \theta*\rho_\varepsilon$ and define

$$S_{n,\varepsilon}(t,x) = \sum_{1\le q+r+s}^{n} \frac{a_{qrs}}{q+1} (w_\varepsilon^{q+1})_x z_\varepsilon^r \theta_\varepsilon^s ,$$

$$S_\varepsilon(t,x) = \sum_{1\le q+r+s}^{\infty} \frac{a_{qrs}}{q+1} (w_\varepsilon^{q+1})_x z_\varepsilon^r \theta_\varepsilon^s .$$

Then using (2.20), (2.22) and the properties of χ, it is obvious that $S_\varepsilon(t,x) \in C((0,\infty);\,\mathcal{M})$ for each $\varepsilon > 0$ and that

$$\begin{aligned} S_\varepsilon(t,x) &= p(w_\varepsilon+z_\varepsilon,\, \theta_\varepsilon)_x - p_u(w_\varepsilon+z_\varepsilon,\, \theta_\varepsilon)\partial_x z_\varepsilon - p_\theta(w_\varepsilon+z_\varepsilon,\, \theta_\varepsilon)\partial_x\theta_\varepsilon + a\partial_x w_\varepsilon \\ &= \{p_u(w_\varepsilon+z_\varepsilon,\, \theta_\varepsilon) + a\}\partial_x w_\varepsilon \end{aligned}$$

holds. Moreover, we can easily see that for each fixed $t > 0$,

$$p(w_\varepsilon + z_\varepsilon,\, \theta_\varepsilon)_x \to p(w+z,\, \theta)_x \quad \text{in } \mathcal{D}^*(R) ,$$

$$p_u(w_\varepsilon + z_\varepsilon,\, \theta_\varepsilon)\partial_x z_\varepsilon \to p_u(w+z,\, \theta)\partial_x z \quad \text{in } \mathcal{D}^*(R) ,$$

$$p_\theta(w_\varepsilon + z_\varepsilon,\, \theta_\varepsilon)\partial_x \theta_\varepsilon \to p_\theta(w+z,\, \theta)\partial_x \theta \quad \text{in } \mathcal{D}^*(R) ,$$

when $\varepsilon \to 0$. Therefore, for each fixed $t > 0$, $S_\varepsilon(t,x) \to \sigma(t,x)$ in $\mathcal{D}^*(R)$. Combining this with the estimate

$$(2.24) \qquad \|S_\varepsilon(t,x)\| \le MK^2(1+t)^{\frac{-1-m}{2}}, \quad \text{for all } \varepsilon > 0,\ t > 0 ,$$

where the constant M is independent of K and t, we conclude that for each fixed $t > 0$, $\sigma(t,x) \in \mathcal{M}$ and $S_\varepsilon(t,x) \to \sigma(t,x)$ in the weak $*$ topology of $\mathcal{M}$, from which (2.23) follows. On the other hand, it is easy to see that

for each fixed $t > 0$ and n, $S_{n,\varepsilon}(t,x) \to S_n(t,x)$ in the weak $*$ topology of $\mathcal{M}$ as $\varepsilon \to 0$. Hence it holds that

$$(2.25)\quad |\langle \sigma(t,x) - S_n(t,x),\ g(x)\rangle| \le \overline{\lim_{\varepsilon\to 0}}\ |\langle S_\varepsilon(t,x) - S_{n,\varepsilon}(t,x),\ g(x)\rangle|$$

$$\le \|g\|_{L^\infty} \sum_{n+1\le q+r+s}^{\infty} |a_{qrs}| K^{q+r+s+1} (1+t)^{-\frac{m}{2}(q+1)-\frac{1}{2}(r+s)}$$

for all $g \in C_0(R)$ and $t > 0$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between C_0 and $\mathcal{M}$. Now the remaining assertion of the lemma follows from (2.25).

(Step II). We shall construct a mapping T from χ into itself. For $(w,z,v,\theta) \in \chi$, $(\tilde{w},\tilde{z},\tilde{v},\tilde{\theta}) = T(w,z,v,\theta)$ is defined by

$$\tilde{w}(t,x) = e^{-at}u_0(x) - \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*\sigma(\tau,x)d\tau$$

$$(2.26)\qquad + \int_0^{\frac{t}{2}} e^{-a(t-\tau)}\{p(w+z,\theta) + aw + az + b\theta\}(\tau,x)d\tau \quad ,$$

where $\sigma(\tau,x)$ is given by (2.22),

$$(2.27)\quad \tilde{z}(t,x) = H_1(t,x)*u_0(x) + G_{12}(t,x)*v_0(x) + G_{13}(t,x)*\theta_0(x)$$

$$- \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*[\{p_u(w+z,\theta) + a\}\partial_x z + \{p_\theta(w+z,\theta) + b\}\partial_x\theta](\tau,x)d\tau$$

$$-\int_0^{\frac{t}{2}} H_5(t-\tau,x)*\{p(w+z,\theta) + aw + az + b\theta\}(\tau,x)d\tau$$

$$-\int_0^{t} G_{13}(t-\tau,x)*[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}\partial_x v](\tau,x)d\tau$$

$$+ \int_0^{t} G_{13}(t-\tau,x)*\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau$$

$$+ \int_0^t G_{13}(t-\tau,x)*[\{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta](\tau,x)d\tau \quad ,$$

$$(2.28)\quad \tilde{v}(t,x) = G_{21}(t,x)*u_0(x) + G_{22}(t,x)*v_0(x) + G_{23}(t,x)*\theta_0(x)$$

$$- \int_0^t G_{22}(t-\tau,x)*\partial_x\{p(w+z,\theta) + aw + az + b\theta\}(\tau,x)d\tau$$

$$- \int_0^t G_{23}(t-\tau,x)*[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}\partial_x v](\tau,x)d\tau$$

$$+ \int_0^t G_{23}(t-\tau,x)*\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau$$

$$+ \int_0^t G_{23}(t-\tau,x)*[\{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta](\tau,x)d\tau \quad ,$$

$$(2.29)\quad \tilde{\theta}(t,x) = G_{31}(t,x)*u_0(x) + G_{32}(t,x)*v_0(x) + G_{33}(t,x)*\theta_0(x)$$

$$- \int_0^t G_{32}(t-\tau,x)*\partial_x\{p(w+z,\theta) + aw + az + b\theta\}(\tau,x)d\tau$$

$$- \int_0^t G_{33}(t-\tau,x)*[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}\partial_x v](\tau,x)d\tau$$

$$+ \int_0^t G_{33}(t-\tau,x)*\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau$$

$$+ \int_0^t G_{33}(t-\tau,x)*[\{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta](\tau,x)d\tau \quad .$$

Since $(w,z,v,\theta) \in \chi$, it is easily seen that $\tilde{w}$, $\tilde{z}$, $\tilde{v}$ and $\tilde{\theta}$ are well-defined as distributions in $((0,\infty) \times R)$ and satisfy the equations:

$$(2.30)\begin{cases} (\tilde{w}+\tilde{z})_t = \tilde{v}_x \\ \tilde{v}_t = a(\tilde{w}+\tilde{z})_x + b\tilde{\theta}_x + \tilde{v}_{xx} - \partial_x\{p(w+z,\theta) + aw + az + b\theta\} \\ \tilde{\theta}_t = d\tilde{v}_x + c\tilde{\theta}_{xx} - \{\dfrac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}v_x + \dfrac{1}{e_\theta(w+z,\theta)}(v_x)^2 \\ \qquad + \{\dfrac{1}{e_\theta(w+z,\theta)} - c\}\theta_{xx} \end{cases},$$

in $\mathcal{D}^*((0,\infty)\times R)$ (see Appendix).

Now we shall prove that $(\tilde{w},\tilde{z},\tilde{v},\tilde{\theta}) \in \chi$. Throughout the remainder of this paper, the constants M will be independent of K and t.

__Lemma 2.3__. $J_1(t,x) \overset{\text{def}}{=} \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*\sigma(\tau,x)d\tau$ satisfies the properties (A) of (Step I), except $w(0,x) = u_0(x)$, with MK^2 in place of K in (2.1) to (2.4) and it holds that $J_1(0,x) = 0$.

__Proof__. Estimates for $\|J_1(t,x)\|$ and $\|\partial_x J_1(t,x)\|$ follow immediately from (2.23) and Lemma 1.7. In order to estimate $\|\partial_t J_1(t,x)\|$ and $\|\partial_t\partial_x J_1(t,x)\|$, we define

$$(2.31)\qquad g_n(t,x) = \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*S_n(\tau,x)d\tau ,$$

where $S_n(\tau,x)$ is given by (2.21). Then on account of (2.25), it is clear that $\|g_n(t,x) - J_1(t,x)\| \to 0$ uniformly on $[0,\infty)$ as $n \to \infty$, from which it follows that

$$\partial_t g_n(t,x) \to \partial_t J_1(t,x) \quad \text{in } \mathcal{D}^*((0,\infty); L^1(R)) ,$$

$$\partial_t\partial_x g_n(t,x) \to \partial_t\partial_x J_1(t,x) \quad \text{in } \mathcal{D}^*((0,\infty)\times R) .$$

Now the proof is completed by the following lemma.

__Lemma 2.4__. For each n, $\partial_t g_n(t,x) \in C((0,\infty); L^1)$, $\partial_t\partial_x g_n(t,x) \in C((0,\infty); \mathcal{M})$ and it holds that

$$(2.32)\qquad \|\partial_t g_n(t,x)\| \le MK^2(1+t)^{-\frac{m}{2}}, \quad \text{for all } t > 0 ,$$

$$(2.33)\qquad \|\partial_t \partial_x g_n(t,x)\| \le MK^2(1+t)^{-\frac{m}{2}}(t^{-\frac{1}{2}} + t^{-\frac{\alpha}{2}}), \quad \text{for all } t > 0 ,$$

where M is independent of K, n and t. Furthermore, as $n,k \to \infty$

$$(2.34)\qquad \|\partial_t g_n(t,x) - \partial_t g_k(t,x)\| \to 0 \quad \text{uniformly on } [0,\infty)$$

and

$$(2.35)\qquad \|\partial_t \partial_x g_n(t,x) - \partial_t \partial_x g_k(t,x)\| \to 0 \quad \text{unformly on each compact subset of } (0,\infty) .$$

__Proof__. Since S_n is a finite sum, we may estimate each term of $g_n(t,x)$. By integrating by parts, we see that

$$(2.36)\ M_{qrs}(t,x) \overset{\text{def}}{=} \partial_t \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*\{(w^{q+1})_x z^r \theta^s\}(\tau,x)d\tau$$

$$= -\partial_t \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*\{w^{q+1}(z^r\theta^s)_x\}(\tau,x)d\tau + \partial_t \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*(w^{q+1}z^r\theta^s)_x(\tau,x)d\tau$$

$$= \frac{1}{2} G_{12}(\tfrac{t}{2},x)*\{w^{q+1}(z^r\theta^s)_x\}(\tfrac{t}{2},x) - \int_{\frac{t}{2}}^{t} \partial_x G_{22}(t-\tau,x)*\{w^{q+1}(z^r\theta^s)_x\}(\tau,x)d\tau$$

$$+ \frac{1}{2} G_{12}(\tfrac{t}{2},x)*(w^{q+1}z^r\theta^s)_x(\tfrac{t}{2},x) - w^{q+1}z^r\theta^s(t,x) + e^{-\frac{at}{2}} w^{q+1}z^r\theta^s(\tfrac{t}{2},x)$$

$$+ a\int_{\frac{t}{2}}^{t} e^{-a(t-\tau)} w^{q+1}z^r\theta^s(\tau,x)d\tau + \int_{\frac{t}{2}}^{t} H_5(t-\tau,x)*(w^{q+1}z^r\theta^s)_\tau(\tau,x)d\tau \quad .$$

Here we have used (1.38) and the fact that $\partial_t G_{12}(t,x) = \partial_x G_{22}(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$ which follows from (1.7) (see Appendix). Applying Lemmas 1.7, 1.9 and the properties of χ, we can derive that

$$(2.37)\quad \begin{cases} M_{qrs}(t,x) \in C((0,\infty); L^1) \\ \|M_{qrs}(t,x)\| \le (q+r+s+1)MK^{q+r+s+1}(1+t)^{-\frac{m}{2}}, \quad \text{for all } t > 0 , \end{cases}$$

where M is a constant independent of t, K, q, r and s. Therefore, we conclude that

$$\partial_t g_n(t,x) = \sum_{1 \leq q+r+s}^{n} \frac{a_{qrs}}{q+1} M_{qrs}(t,x) \in C((0,\infty); L^1)$$

and, by recalling (2.19),

$$(2.38) \qquad \|\partial_t g_n(t,x)\| \leq \sum_{1 \leq q+r+s}^{n} \frac{q+r+s+1}{q+1} |a_{qrs}| M\, K^{q+r+s+1} (1+t)^{-\frac{m}{2}}$$

$$\leq MK^2 (1+t)^{-\frac{m}{2}}, \quad \text{for all } t > 0 \text{ and } n \geq 1 \quad ,$$

where M denotes different constants independent of K, t and all the dummy indices. From the estimate

$$(2.39) \quad \|\partial_t g_n(t,x) - \partial_t g_k(t,x)\| \leq \sum_{k+1 \leq q+r+s}^{n} \frac{q+r+s+1}{q+1} |a_{qrs}| M\, K^{q+r+s+1} (1+t)^{-\frac{m}{2}}$$

for all $t > 0$ and $n \geq k+1$, we get (2.34). Next, we define

$$(2.40) \qquad M^{\varepsilon}_{qrs}(t,x) = \partial_t \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x) * \{(w_\varepsilon^{q+1})_x z_\varepsilon^r \theta_\varepsilon^s\}(\tau,x)\,d\tau \quad ,$$

where $w_\varepsilon = w*\rho_\varepsilon$, $z_\varepsilon = z*\rho_\varepsilon$, $\theta_\varepsilon = \theta*\rho_\varepsilon$. Then using (2.36), we have

$$(2.41) \quad \partial_x M^{\varepsilon}_{qrs}(t,x) = \frac{1}{2}\, \partial_x G_{12}(\tfrac{t}{2},x) * \{w_\varepsilon^{q+1} (z_\varepsilon^r \theta_\varepsilon^s)_x\}(\tfrac{t}{2},x)$$

$$- \int_{\frac{t}{2}}^{t} \partial_x G_{22}(t-\tau,x) * \{(w_\varepsilon^{q+1})_x (z_\varepsilon^r \theta_\varepsilon^s)_x + w_\varepsilon^{q+1} (z_\varepsilon^r \theta_\varepsilon^s)_{xx}\}(\tau,x)\,d\tau$$

$$+ \frac{1}{2}\, \partial_x G_{12}(\tfrac{t}{2},x) * (w_\varepsilon^{q+1} z_\varepsilon^r \theta_\varepsilon^s)_x (\tfrac{t}{2},x)$$

$$- (w_\varepsilon^{q+1} z_\varepsilon^r \theta_\varepsilon^s)_x (t,x) + e^{-\frac{at}{2}} (w_\varepsilon^{q+1} z_\varepsilon^r \theta_\varepsilon^s)_x (\tfrac{t}{2},x)$$

$$+ a \int_{\frac{t}{2}}^{t} e^{-a(t-\tau)} (w_\varepsilon^{q+1} z_\varepsilon^r \theta_\varepsilon^s)_x (\tau,x)\,d\tau$$

$$+\int_{\frac{t}{2}}^{t} \partial_x H_5(t-\tau,x)*(w_\varepsilon^{q+1} z_\varepsilon^r \theta_\varepsilon^s)_\tau(\tau,x)d\tau \quad .$$

In order to estimate the last integral, we need to observe that (1.41) implies

$$\|\partial_x H_5(t,x)\| \le Mt^{-\frac{3}{4}}, \quad \text{for all} \quad t > 0 \quad . \tag{2.42}$$

Combining this with (2.19), the properties of χ and Lemmas 1.7, 1.9, we obtain

$$\partial_x M_{qrs}^\varepsilon(t,x) \in C((0,\infty); L^1) \quad , \tag{2.43}$$

$$\|\partial_x M_{qrs}^\varepsilon(t,x)\| \le (q+r+s+1)^2 M\, K^{q+r+s+1}(1+t)^{-\frac{m}{2}}(t^{-\frac{1}{2}} + t^{-\frac{\alpha}{2}}), \quad \text{for all } t > 0 \quad , \tag{2.44}$$

$$\|\partial_x M_{qrs}^\varepsilon(t_1,x) - \partial_x M_{qrs}^\varepsilon(t_2,x)\| \le \rho(t_1,t_2)(q+r+s+1)^3 M\, K^{q+r+s}, \text{ for all } t_1,t_2 > 0, \tag{2.45}$$

where M is a constant independent of t, q, r, s, ε, K, and $\rho(t_1,t_2)$ is a function of $t_1,t_2 > 0$, independent of q, r, s, ε, which tends to zero as $t_2 \to t_1 > 0$. Comparing (2.36) with its analog for $M_{qrs}^\varepsilon(t,x)$ and using the fact that for each $t > 0$,

$$z_\varepsilon \to z,\ \theta_\varepsilon \to \theta \quad \text{in} \quad C_0(R) \quad ,$$

$$w_\varepsilon \to w,\ \partial_x z_\varepsilon \to \partial_x z,\ \partial_x \theta_\varepsilon \to \partial_x \theta \quad \text{in} \quad L^1(R) \quad ,$$

$$\partial_t w_\varepsilon \to \partial_t w,\ \partial_t z_\varepsilon \to \partial_t z,\ \partial_t \theta_\varepsilon \to \partial_t \theta \quad \text{in} \quad L^1(R) \quad ,$$

$$w_\varepsilon^q \to w^q \quad \text{weak * in} \quad L^\infty(R) \quad \text{for all positive integer } q \quad ,$$

$$\partial_t w,\ \partial_x z,\ \partial_x \theta \in L^\infty(R) \quad ,$$

it is easily seen that $M_{qrs}^\varepsilon(t,x)$ converges to $M_{qrs}(t,x)$ in $\mathcal{D}^*(R)$ for each $t > 0$, which implies that $\partial_x M_{qrs}^\varepsilon(t,x)$ converges to $\partial_x M_{qrs}(t,x)$ in $\mathcal{D}^*(R)$ for each $t > 0$. Combining this with (2.43), (2.44), (2.45), we derive that $\partial_x M_{qrs}(t,x)$ is the weak * limit of $\partial_x M_{qrs}^\varepsilon(t,x)$ in $\mathcal{M}$ for each $t > 0$, and that

$$\|\partial_x M_{qrs}(t,x)\| \le (q+r+s+1)^2 M\, K^{q+r+s+1}(1+t)^{-\frac{m}{2}}(t^{-\frac{1}{2}} + t^{-\frac{\alpha}{2}}), \text{ for all } t > 0 \quad , \tag{2.46}$$

(2.47) $\|\partial_x M_{qrs}(t_1,x) - \partial_x M_{qrs}(t_2,x)\| \leq \rho(t_1,t_2)(q+r+s+1)^3 M\, K^{q+r+s}$, for all $t_1,t_2 > 0$,

from which it follows that

(2.48) $$\partial_x M_{qrs}(t,x) \in C((0,\infty); \mathcal{M}) \quad .$$

Now it is obvious that

(2.49) $$\partial_t \partial_x g_n(t,x) = \sum_{1 \leq q+r+s}^{n} \frac{a_{qrs}}{q+1} \partial_x M_{qrs}(t,x) \in C((0,\infty); \mathcal{M}) \quad ,$$

and (2.33), (2.35) hold.

<u>Lemma 2.5</u>. $J_2(t,x) \overset{\text{def}}{=} \int_0^{\frac{t}{2}} e^{-a(t-\tau)}\{p(w+z,\theta) + aw + az + b\theta\}(\tau,x)d\tau$ has the same properties as were stated in Lemma 2.3.

<u>Proof</u>. Proceeding as in Lemma 2.2, it is easy to observe that

(2.50) $$\begin{cases} p(w+z,\theta) + aw + az + b\theta \in C((0,\infty); L^1) \quad , \\ \|p(w+z,\theta) + aw + az + b\theta\| \leq MK^2(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad , \end{cases}$$

and

(2.51) $$\begin{cases} p(w+z,\theta)_x + aw_x + az_x + b\theta_x \in C((0,\infty); \mathcal{M}) \quad , \\ \|p(w+z,\theta)_x + aw_x + az_x + b\theta_x\| \leq MK^2(1+t)^{-1}, \quad \text{for all} \quad t > 0 \quad , \end{cases}$$

which yield the result.

Before proceeding to get other estimates, we note the following fact:

<u>Lemma 2.6</u>. Suppose $g(\cdot,\cdot) \in C^1(R \times R)$, $g(0,0) = 0$ and $|Dg(\cdot,\cdot)|$ is bounded by the constant L. Let $h_1(t,x)$, $h_2(t,x)$ and $h_3(t,x)$ belong to $C((0,\infty); L^1 \cap BV)$. Then, it holds that

(2.52) $$\partial_x\{g(h_1(t,x), h_2(t,x))h_3(t,x)\} \in C((0,\infty); \mathcal{M})$$

and

(2.53) $$\|\partial_x\{g(h_1(t,x), h_2(t,x))h_3(t,x)\}\| \leq$$
$$\leq \frac{3}{2} L\|\partial_x h_3(t,x)\|(\|\partial_x h_1(t,x)\| + \|\partial_x h_2(t,x)\|) \text{ for all } t > 0 \quad .$$

Proof. Regularizing h_1, h_2, h_3 with respect to x and using the convergence argument in Lemma 2.2, we can obtain the result.

Lemma 2.7. $J_3(t,x) \overset{\text{def}}{=} \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*[\{p_u(w+z,\theta) + a\}z_x + \{p_\theta(w+z,\theta) + b\}\theta_x](\tau,x)d\tau$ satisfies the properties (B) of (Step I) with MK^2 in place of K in (2.5) to (2.7).

Proof. The proof follows immediately from the properties of χ and Lemmas 1.7, 2.6.

Lemma 2.8. $J_4(t,x) \overset{\text{def}}{=} \int_0^{\frac{t}{2}} H_5(t-\tau,x)*[p(w+z,\theta) + aw + az + b\theta](\tau,x)d\tau$ satisfies the same properties as were stated in Lemma 2.7.

Proof. It suffices to combine Lemma 1.7 with (2.50), (2.51).

Lemma 2.9. $J_5(t,x) \overset{\text{def}}{=} \int_0^t G_{13}(t-\tau,x)*[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}v_x](\tau,x)d\tau$ satisfies the same properties as were stated in Lemma 2.7.

Proof. Since $\frac{p_\theta(\cdot,\cdot)}{e_\theta(\cdot,\cdot)}$ is an analytic function in a neighborhood of $(0,0)$ and $-\frac{p_\theta(0,0)}{e_\theta(0,0)}\bar{\theta} = d$, we can write

$$\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d = a_{10}(w+z) + a_{01}\theta + \sum_{2\leq q+r}^{\infty} a_{qr}(w+z)^q\theta^r \tag{2.54}$$

if $|w|, |z|, |\theta| \leq K$ (recall the condition (2.19)). Break J_5 into $J_{5,1} + J_{5,2}$, where

$$J_{5,1}(t,x) = \int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}v_x](\tau,x)d\tau \ , \tag{2.55}$$

$$J_{5,2}(t,x) = \int_{\frac{t}{2}}^{t} G_{13}(t-\tau,x)*[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}v_x](\tau,x)d\tau \ . \tag{2.56}$$

Using the property (C) of (Step I), we see that, for each $t > 0$,

$$(2.57)\quad \int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{(w+z)v_x\}(\tau,x)d\tau = \lim_{\varepsilon\to 0}\int_\varepsilon^{\frac{t}{2}} G_{13}(t-\tau,x)*\{(w+z)v_x\}(\tau,x)d\tau$$

$$= \lim_{\varepsilon\to 0}\int_\varepsilon^{\frac{t}{2}} G_{13}(t-\tau,x)*\{(w+z)(w+z)_\tau\}(\tau,x)d\tau$$

$$= \frac{1}{2} G_{13}(\tfrac{t}{2},x)*(w+z)^2(\tfrac{t}{2},x) - \frac{1}{2} G_{13}(t,x)*(w+z)^2(0,x)$$

$$+ \frac{1}{2}\lim_{\varepsilon\to 0}\int_\varepsilon^{\frac{t}{2}} \partial_t G_{13}(t-\tau,x)*(w+z)^2(\tau,x)d\tau \quad .$$

But $\partial_t G_{13}(t,x) = \partial_x G_{23}(t,x)$ in $\mathcal{D}^*((0,\infty)\times R)$ and hence, by virtue of Lemmas 1.8, 1.10, we obtain

$$(2.58)\qquad \int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{(w+z)v_x\}(\tau,x)d\tau \in C([0,\infty);\, L^1)$$

and, assuming $\|u_0\| + \|\partial_x u_0\| \leq K$ (which will be fulfilled by (2.200)),

$$(2.59)\qquad \left\|\int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{(w+z)v_x\}(\tau,x)d\tau\right\| \leq MK^2, \quad \text{for all } t > 0 \quad .$$

Next we have

$$(2.60)\quad \int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\theta v_x\}(\tau,x)d\tau = \lim_{\varepsilon\to 0}\int_\varepsilon^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\tfrac{1}{d}\,\theta\theta_\tau\}(\tau,x)d\tau$$

$$+ \lim_{\varepsilon\to 0}\int_\varepsilon^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\theta(v_x - \tfrac{1}{d}\,\theta_\tau)\}(\tau,x)d\tau \quad .$$

The L^1-norm of the first integral on the right hand side can be estimated by integration by parts as in the derivation of (2.58), (2.59), and the L^1-norm of the second integral can be estimated directly with the aid of (2.17); we obtain

$$\int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\theta v_x\}(\tau,x)d\tau \in C([0,\infty);\ L^1) \tag{2.61}$$

and, assuming $\|\theta_0\| + \|\partial_x\theta_0\| \le K$ (which will also be fulfilled by (2.200)),

$$\left\|\int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\theta v_x\}(\tau,x)d\tau\right\| \le MK^2, \quad \text{for all} \quad t \ge 0 \quad . \tag{2.62}$$

Noticing that

$$\sum_{2<q+r}^{\infty} a_{qr}(w+z)^q\theta^r v_x \in C((0,\infty);\ L^1) \quad , \tag{2.63}$$

$$\left\|\sum_{2<q+r}^{\infty} a_{qr}(w+z)^q\theta^r v_x\right\| \le MK^3(1+t)^{-\frac{3}{2}}, \quad \text{for all} \quad t > 0 \quad , \tag{2.64}$$

we derive that

$$\int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\sum_{2<q+r}^{\infty} a_{qr}(w+z)^q\theta^r v_x\}(\tau,x)d\tau \in C([0,\infty);\ L^1) \tag{2.65}$$

and, by (2.19),

$$\left\|\int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\sum_{2<q+r}^{\infty} a_{qr}(w+z)^q\theta^r v_x\}(\tau,x)d\tau\right\| \le MK^2, \quad \text{for all} \quad t \ge 0 \quad . \tag{2.66}$$

Hence, we conclude that

$$J_{5,1}(t,x) \in C([0,\infty);\ L^1),\ J_{5,1}(0,x) = 0 \tag{2.67}$$

and

$$\|J_{5,1}(t,x)\| \le MK^2, \quad \text{for all} \quad t \ge 0 \quad . \tag{2.68}$$

Next we can directly obtain

$$J_{5,2}(t,x) \in C([0,\infty);\ L^1),\ J_{5,2}(0,x) = 0 \tag{2.69}$$

and

$$\|J_{5,2}(t,x)\| \le MK^2, \quad \text{for all} \quad t \ge 0 \quad , \tag{2.70}$$

from

$$(2.71)\qquad \{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar\theta+\theta)+d\}v_x(t,x)\in C((0,\infty);L^1)\quad,$$

$$(2.72)\qquad \|\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar\theta+\theta)+d\}v_x(t,x)\|\le MK^2(1+t)^{-1},\quad \text{for all}\quad t>0\quad .$$

Thus, (2.5) has been proved with K replaced by MK^2. In order to estimate the L^1-norm of $\partial_xJ_5=\partial_xJ_{5,1}+\partial_xJ_{5,2}$, it suffices to replace $G_{13}(t-\tau,x)$ by $\partial_xG_{13}(t-\tau,x)$ for both $\partial_xJ_{5,1}$ and $\partial_xJ_{5,2}$. However, we note that for the case $t\le 1$, $\|\partial_xJ_{5,1}\|$ can be estimated directly without going through the lengthy procedure as was done for $\|J_{5,1}\|$. Finally, we will esti
$\|\partial_{xx}J_5\|$. By virtue of (2.19) and Lemma 2.6, we have

$$(2.73)\qquad \partial_x[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar\theta+\theta)+d\}v_x](t,x)\in C((0,\infty);M)$$

and

$$(2.74)\qquad \|\partial_x[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar\theta+\theta)+d\}v_x](t,x)\|\le MK^2t^{-\frac{1}{2}}(1+t)^{\frac{-1-\alpha}{2}},\quad \text{for all}\quad t>0\quad ,$$

from which it follows that

$$(2.75)\qquad \partial_{xx}J_{5,2}(t,x)=\int_{\frac{t}{2}}^{t}\partial_xG_{13}(t-\tau,x)*\partial_x[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar\theta+\theta)+d\}v_x](\tau,x)d\tau\in C((0,\infty);L^1)\quad ,$$

$$(2.76)\qquad \partial_{xx}J_{5,1}(t,x)=\int_{0}^{\frac{t}{2}}\partial_{xx}G_{13}(t-\tau,x)*[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar\theta+\theta)+d\}v_x](\tau,x)d\tau\in C((0,\infty);L^1)$$

and

$$(2.77)\qquad \|\partial_{xx}J_{5,2}(t,x)\|,\ \|\partial_{xx}J_{5,1}(t,x)\|\le MK^2t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}},\quad \text{for all}\quad t>0\quad .$$

Lemma 2.10. $J_6(t,x) \overset{\text{def}}{=} \int_0^t G_{13}(t-\tau,x)*\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau$ has the same properties as were stated in Lemma 2.7.

Proof. Using the properties of χ, we see that

$$(2.78)\qquad \begin{cases} \frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2 \in C((0,\infty);\, L^1) \\ \\ \|\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\| \le MK^2 t^{-\frac{1}{2}}(1+t)^{\frac{-1-\alpha}{2}}, \quad \text{for all } t > 0 \end{cases},$$

and, applying Lemma 2.6 with some modification,

$$(2.79)\qquad \begin{cases} \partial_x\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\} \in C((0,\infty);\, M) \\ \\ \|\partial_x\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}\| \le MK^2 t^{-1}(1+t)^{-\alpha}, \quad \text{for all } t > 0 \end{cases}.$$

From (2.78), (2.79), we can easily get (2.5) with K replaced by MK^2. Let us define

$$(2.80)\qquad J_{6,1}(t,x) = \int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau$$

and

$$(2.81)\qquad J_{6,2}(t,x) = \int_{\frac{t}{2}}^{t} G_{13}(t-\tau,x)*\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau \quad .$$

Then, using the properties of $G_{13}(t,x)$, it is easily seen that

$$(2.82)\qquad \partial_x J_{6,1}(t,x),\ \partial_x J_{6,2}(t,x) \in C([0,\infty);\, L^1)$$

and

$$(2.83)\qquad \|\partial_x J_{6,1}(t,x)\|,\ \|\partial_x J_{6,2}(t,x)\| \le MK^2(1+t)^{-\frac{1}{2}}, \quad \text{for all } t > 0 \quad .$$

Observing that

$$(2.84)\qquad \partial_{xx}J_{6,1}(t,x) = \int_0^{\frac{t}{2}} \partial_{xx}G_{13}(t-\tau,x)*\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau \quad ,$$

$$(2.85)\qquad \partial_{xx}J_{6,2}(t,x) = \int_{\frac{t}{2}}^t \partial_x G_{13}(t-\tau,x)*\partial_x\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau \quad ,$$

we can derive that

$$(2.86)\qquad \partial_{xx}J_{6,1}(t,x),\ \partial_{xx}J_{6,2}(t,x) \in C((0,\infty);\ L^1)$$

and

$$(2.87)\qquad \|\partial_{xx}J_{6,1}(t,x)\|,\ \|\partial_{xx}J_{6,2}(t,x)\| \le MK^2 t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad .$$

<u>Lemma 2.11</u>. $J_7(t,x) \overset{\text{def}}{=} \int_0^t G_{13}(t-\tau,x)*[\{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta](\tau,x)d\tau$ satisfies the same properties as were stated in Lemma 2.7.

<u>Proof</u>. First, observe that

$$(2.88)\qquad \{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta(t,x) \in C((0,\infty);\ L^1)$$

and

$$(2.89)\qquad \|\{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta(t,x)\| \le MK^2 t^{-\frac{1}{2}}(1+t)^{\frac{-1-\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad .$$

Proceeding similarly to the proof of Lemma 2.10, we can easily verify that $J_7(t,x)$, $\partial_x J_7(t,x)$ satisfy the required properties. Next, recalling the fact that $\partial_{xx}G_{13}(t,x) = \frac{b}{c}e^{-at}\delta(x) + H_8(t,x)$, where $H_8(t,x) \in C((0,\infty);\ L^1)$ with the estimate (1.53), we can write

$$(2.90)\qquad \begin{aligned}\partial_{xx}J_7(t,x) &= \int_{\frac{t}{2}}^t \partial_{xx}G_{13}(t-\tau,x)*[\{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta](\tau,x)d\tau \\ &+ \int_0^{\frac{t}{2}} \partial_{xx}G_{13}(t-\tau,x)*[\{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta](\tau,x)d\tau\end{aligned}$$

and estimate these two inegrals separately. Using

$$(2.91)\qquad \|H_8(t,x)\| \le Mt^{-\frac{1}{2}}, \quad \text{for all } t > 0 \quad \text{(which follows from (1.53))} ,$$

for the first integral and

$$(2.92)\qquad \|H_8(t,x)\| \le Mt^{-1}, \quad \text{for all } t > 0 ,$$

for the second integral, we obtain

$$(2.93)\qquad \partial_{xx}J_7(t,x) \in C((0,\infty); L^1)$$

and

$$(2.94)\qquad \|\partial_{xx}J_7(t,x)\| \le MK^2t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 .$$

<u>Lemma 2.12</u>. $J_8(t,x) \overset{def}{=} \int_0^t G_{22}(t-\tau,x)*\partial_x\{p(w+z,\theta) + aw + az + b\theta\}(\tau,x)d\tau$ satisfies the properties (D) of (Step I) with K replaced by MK^2, except $v(0,x) = v_0$ and (2.11). In addition, $J_8(0,x) = 0$.

<u>Proof</u>. Breaking $J_8(t,x)$ into two parts by

$$(2.95)\qquad \begin{aligned} J_8(t,x) &= \int_{\frac{t}{2}}^t G_{22}(t-\tau,x)*\partial_x\{p(w+z,\theta) + aw + az + b\theta\}(\tau,x)d\tau \\ &+ \int_0^{\frac{t}{2}} \partial_xG_{22}(t-\tau,x)*\{p(w+z,\theta) + aw + az + b\theta\}(\tau,x)d\tau , \end{aligned}$$

we can easily find (2.8), (2.9) with K replaced by MK^2 with the aid of (2.50) and (2.51), which, combined with the dominated convergence theorem, also yield

$$(2.96)\qquad J_8(t,x) \in C([0,\infty); L^1),\ J_8(0,x) = 0$$

$$(2.97)\qquad \partial_xJ_8(t,x) \in C((0,\infty); L^1) .$$

Since $\|\partial_{xx}G_{22}(t-\tau,x)\|$ is not integrable over $(0,t)$, it is rather complicated to estimate $\|\partial_{xx}J_8(t,x)\|$. First, recalling (0.10) and (2.19), we write

$$(2.98)\qquad p(w+z,\theta) + aw + az + b\theta = \sum_{2\le q+r+s}^{\infty} b_{qrs}\, w^qz^r\theta^s$$

and define

(2.99) $$F_n(t,x) = \sum_{2\leq q+r+s}^{n} b_{qrs}\, w^q z^r \theta^s \quad ,$$

(2.100) $$J_{8,n}(t,x) = \int_0^t G_{22}(t-\tau,x)*\partial_x F_n(\tau,x)d\tau \quad .$$

Then, $F_n(t,x)$ converges to $\{p(w+z,\theta) + aw + az + b\theta\}(t,x)$ in $L^1(R)$ uniformly on $(0,\infty)$ as $n \to \infty$. Therefore, $\partial_{xx}J_{8,n}(t,x)$ converges to $\partial_{xx}J_8(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$. Since $F_n(t,x)$ is a finite series, we can estimate $\partial_{xx}J_{8,n}(t,x)$ term by term. Set

(2.101) $$Q_{qrs}(t,x) = \partial_{xx} \int_{\frac{t}{2}}^t G_{22}(t-\tau,x)*\partial_x\{w^q z^r \theta^s\}(\tau,x)d\tau \quad .$$

Assuming Lemma 2.13 which will be proved subsequently, we see that

(2.102) $$\begin{aligned} Q_{qrs}(t,x) = &-\partial_x\{w^q z^r \theta^s\}(t,x) + G_{22}(\tfrac{t}{2},x)*\partial_x\{w^q z^r \theta^s\}(\tfrac{t}{2},x) \\ &+ \int_{\frac{t}{2}}^t \partial_x G_{22}(t-\tau,x)*\partial_\tau\{w^q z^r \theta^s\}(\tau,x)d\tau \\ &-a \int_{\frac{t}{2}}^t \partial_x G_{12}(t-\tau,x)*\partial_x\{w^q z^r \theta^s\}(\tau,x)d\tau \\ &-b \int_{\frac{t}{2}}^t \partial_x G_{32}(t-\tau,x)*\partial_x\{w^q z^r \theta^s\}(\tau,x)d\tau \end{aligned}$$

holds in $\mathcal{D}^*((0,\infty) \times R)$. Considering each term of the right-hand side, we deduce that, for $q \geq 1$,

(2.103) $$Q_{qrs}(t,x) \in C((0,\infty); M)$$

and

(2.104) $$\|Q_{qrs}(t,x)\| \leq (q+r+s)MK^{q+r+s}t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 \quad ,$$

where M is independent of q, r, s, K and t. For the case $q = 0$, $r+s \geq 2$, we note that

(2.105) $$\begin{cases} \partial_{xx}\{z^r\theta^s\}(t,x) \in C((0,\infty); M) \\ \|\partial_{xx}\{z^r\theta^s\}(t,x)\| \le (r+s)(r+s-1)MK^{r+s}t^{-\frac{1}{2}}(1+t)^{\frac{-1-\alpha}{2}}, \quad \text{for all } t > 0, \end{cases}$$

where M is independent of r, s, K and t, and use the formula

(2.106) $$Q_{ors}(t,x) = \int_{\frac{t}{2}}^{t} \partial_x G_{22}(t-\tau,x) * \partial_{xx}\{z^r\theta^s\}(\tau,x)\,d\tau$$

to find that

(2.107) $$Q_{ors}(t,x) \in C((0,\infty); L^1)$$

(2.108) $$\|Q_{ors}(t,x)\| \le (r+s)(r+s-1)MK^{r+s}t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 .$$

Next, set

(2.109) $$R_{qrs}(t,x) = \partial_{xx}\int_0^{\frac{t}{2}} G_{22}(t-\tau,x) * \partial_x\{w^q z^r\theta^s\}(\tau,x)\,d\tau .$$

Recalling that

(2.110) $$\begin{cases} \partial_{xx}G_{22}(t,x) \in C((0,\infty); M) \\ \|\partial_{xx}G_{22}(t,x)\| \le Mt^{-1}, \quad \text{for all } t > 0, \end{cases}$$

we get, for the case $q+r+s \ge 2$,

(2.111) $$R_{qrs}(t,x) \in C((0,\infty); M) ,$$

(2.112) $$\|R_{qrs}(t,x)\| \le (q+r+s)MK^{q+r+s}t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 .$$

From the properties of Q_{qrs}, R_{qrs} and the fact that

$\sum_{2\le q+r+s}^{\infty} b_{qrs}(q+r+s)^2(\frac{\nu}{2})^{q+r+s}$ is an absolutely convergent series, it follows that

(2.113) $$\partial_{xx}J_{8,n}(t,x) \in C((0,\infty); M)$$

(2.114) $$\|\partial_{xx}J_{8,n}(t,x)\| \le MK^2 t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0,\ n \ge 2 ,$$

(2.115) $$\|\partial_{xx}J_{8,n}(t,x) - \partial_{xx}J_{8,k}(t,x)\| \to 0 \quad \text{uniformly on each compact subset}$$

of $(0,\infty)$ as $n, k \to \infty$.

Hence, we conclude that

(2.116) $$\partial_{xx}J_8(t,x) \in C((0,\infty); M) ,$$

(2.117) $$\|\partial_{xx}J_8(t,x)\| \le MK^2 t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 .$$

To complete our argument, we shall present:

Lemma 2.13. Let $g(t,x) \in C((0,\infty); L^1 \cap BV)$, $\partial_t g(t,x) \in C((0,\infty); L^1)$, and set $Q(t,x) = \int_{\frac{t}{2}}^{t} G_{22}(t-\tau,x)*\partial_x g(\tau,x)d\tau$. Then, it holds that

(2.118) $$\begin{aligned}\partial_{xx}Q(t,x) = &-\partial_x g(t,x) + G_{22}(\tfrac{t}{2},x)*\partial_x g(\tfrac{t}{2},x)\\ &+ \int_{\frac{t}{2}}^{t} \partial_x G_{22}(t-\tau,x)*\partial_\tau g(\tau,x)d\tau\\ &-a \int_{\frac{t}{2}}^{t} \partial_x G_{12}(t-\tau,x)*\partial_x g(\tau,x)d\tau\\ &-b \int_{\frac{t}{2}}^{t} \partial_x G_{32}(t-\tau,x)*\partial_x g(\tau,x)d\tau \quad \text{in } \mathcal{D}^*((0,\infty) \times R) .\end{aligned}$$

Proof. Define

(2.119) $$Q_\varepsilon(t,x) = \int_{\frac{t}{2}}^{\max(\frac{t}{2},t-\varepsilon)} G_{22}(t-\tau,x)*\partial_x g(\tau,x)d\tau .$$

Then, it is easily seen that $Q_\varepsilon(t,x) \to Q(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$ and hence, $\partial_{xx}Q_\varepsilon(t,x) \to \partial_{xx}Q(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$. In the mean time, we have, for $0 < \varepsilon < \frac{t}{2}$,

$$(2.120)\quad \partial_{xx}Q_\varepsilon(t,x) = \int_{\frac{t}{2}}^{t-\varepsilon} \partial_{xx}G_{22}(t-\tau,x)*\partial_x g(\tau,x)\,d\tau$$

$$= -a \int_{\frac{t}{2}}^{t-\varepsilon} \partial_x G_{12}(t-\tau,x)*\partial_x g(\tau,x)\,d\tau - b \int_{\frac{t}{2}}^{t-\varepsilon} \partial_x G_{32}(t-\tau,x)*\partial_x g(\tau,x)\,d\tau$$

$$+ \int_{\frac{t}{2}}^{t-\varepsilon} \partial_t G_{22}(t-\tau,x)*\partial_x g(\tau,x)\,d\tau \quad ,$$

which follows from the identity

$$\partial_{xx}G_{22}(t,x) = -a\partial_x G_{12}(t,x) - b\partial_x G_{32}(t,x) + \partial_t G_{22}(t,x) \quad \text{in } \mathcal{D}^*((0,\infty)\times R) \quad .$$

But we see that

$$(2.121)\quad \int_{\frac{t}{2}}^{t-\varepsilon} \partial_t G_{22}(t-\tau,x)*\partial_x g(\tau,x)\,d\tau = G_{22}(\tfrac{t}{2},x)*\partial_x g(\tfrac{t}{2},x) - G_{22}(\varepsilon,x)*\partial_x g(t-\varepsilon,x) + \int_{\frac{t}{2}}^{t-\varepsilon} \partial_x G_{22}(t-\tau,x)*\partial_\tau g(\tau,x)\,d\tau$$

and

$$(2.122)\quad G_{22}(\varepsilon,x)*\partial_x g(t-\varepsilon,x) = H_{14}(\varepsilon,x)*\partial_x g(t,x) + F_\xi^{-1}\{e^{-\varepsilon\xi^2} i\xi\,\hat{g}(t,\xi)\} + G_{22}(\varepsilon,x)*\{\partial_x g(t-\varepsilon,x) - \partial_x g(t,x)\} \quad ,$$

which follows from Lemma 1.9. Using the fact that

$$(2.123)\quad \|H_{14}(\varepsilon,x)\| \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and that for each fixed $t > 0$,

$$(2.124)\quad \|\partial_x g(t-\varepsilon,x) - \partial_x g(t,x)\| \to 0 \quad \text{as} \quad \varepsilon \to 0$$

$$(2.125)\quad e^{-\varepsilon\xi^2} i\xi\,\hat{g}(t,\xi) \to i\xi\,\hat{g}(t,\xi) \quad \text{in tempered distribution as} \quad \varepsilon \to 0 \quad ,$$

we can easily obtain (2.118) by letting ε tend to zero.

We proceed to estimate the remaining integrals. Let us define

$$J_9(t,x) = \int_0^t G_{23}(t-\tau,x)*[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}\partial_x v](\tau,x)d\tau \quad , \tag{2.126}$$

$$J_{10}(t,x) = \int_0^t G_{23}(t-\tau,x)*\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau \quad , \tag{2.127}$$

$$J_{11}(t,x) = \int_0^t G_{23}(t-\tau,x)*[\{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta](\tau,x)d\tau \quad . \tag{2.128}$$

Then, proceeding analogously to the proof of Lemmas 2.9 to 2.11, we can obtain the following result:

__Lemma 2.14__. $J_9(t,x)$, $J_{10}(t,x)$ and $J_{11}(t,x)$ satisfy the same properties as were stated in Lemma 2.12.

__Lemma 2.15__. $J_{12}(t,x) \stackrel{\text{def}}{=} \int_0^t G_{32}(t-\tau,x)*\partial_x\{p(w+z,\theta) + aw + az + b\theta\}(\tau,x)d\tau$ satisfies the properties (E) of (Step I) with K replaced by MK^2, except $\theta(0,x) = \theta_0$, (2.16) and (2.17). In addition, $J_{12}(0,x) = 0$ holds.

__Proof__. The assertions concerning $J_{12}(t,x)$, $\partial_x J_{12}(t,x)$ and $\partial_{xx}J_{12}(t,x)$ can be verified by the method of proof of Lemma 2.12. But $G_{32}(t,x)$ behaves better than $G_{22}(t,x)$ and hence, we can estimate $\|\partial_{xx}J_{12}(t,x)\|$ more directly. Note that (1.81) yields

$$\|\partial_{xx}G_{32}(t,x)\| \le Mt^{-1+\beta}, \quad \text{for all } t > 0, \text{ all } 0 < \beta < \frac{1}{2} \quad . \tag{2.129}$$

Combining this with (2.51), we can easily derive that

$$\partial_{xx}J_{12}(t,x) \in C((0,\infty); L^1) \tag{2.130}$$

$$\|\partial_{xx}J_{12}(t,x)\| \le MK^2 t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 \quad . \tag{2.131}$$

It remains to estimate $|||\partial_{xx}J_{12}(t,x)|||_\alpha$. Using (1.81), (1.82) and (1.133), we conclude that

$$\partial_{xx}G_{23}(t,x) \in C((0,\infty); \Lambda_\alpha^{1,\infty}) \quad , \tag{2.132}$$

$$|||\partial_{xx}G_{23}(t,x)|||_\alpha \leq Mt^{-\frac{\alpha}{2}}(t+t^{\frac{1}{2}})^{-1}$$

(2.133)

$$\leq Mt^{-1+\frac{\alpha}{3}}, \quad \text{by} \quad 0 < \alpha \leq \frac{1}{3} \ .$$

Therefore, we have

$$|||\partial_{xx}J_{12}(t,x)|||_\alpha \leq \int_0^t |||\partial_{xx}G_{23}(t-\tau,x)|||_\alpha \|\partial_x\{p(w+z,\theta)+aw+az+b\theta\}(\tau,x)\|d\tau$$

(2.134)

$$\leq MK^2 \int_0^t (t-\tau)^{-1+\frac{\alpha}{3}}(1+\tau)^{-1}d\tau$$

$$\leq MK^2 t^{\frac{-1-\alpha}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \ .$$

(Here we have used again the fact that $0 < \alpha \leq \frac{1}{3}$.)

<u>Lemma 2.16</u>. $J_{13}(t,x) \overset{\text{def}}{=} \int_0^t G_{33}(t-\tau,x)*[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}\partial_x v](\tau,x)d\tau$ satisfies the same properties as were stated in Lemma 2.15.

<u>Proof</u>. Using Lemma 1.11 and the identity

(2.135) $\quad \partial_t G_{33}(t,x) = c\partial_{xx}G_{33}(t,x) + d\partial_x G_{23}(t,x) \quad \text{in} \quad \mathcal{D}^*((0,\infty) \times R)$,

we can proceed similarly to the proof of Lemma 2.9 to arrive at

(2.136)
$$\begin{cases} J_{13}(t,x) \in C([0,\infty);\ L^1), \ J_{13}(0,x) = 0 \\ \|J_{13}(t,x)\| \leq MK^2, \quad \text{for all} \quad t > 0 \ , \end{cases}$$

(2.137)
$$\begin{cases} \partial_x J_{13}(t,x) \in C((0,\infty);\ L^1) \\ \|\partial_x J_{13}(t,x)\| \leq MK^2(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \ , \end{cases}$$

$$(2.138)\quad \begin{cases} \partial_{xx}J_{13}(t,x) \in C((0,\infty);\ L^1) \\ \|\partial_{xx}J_{13}(t,x)\| \leq MK^2 t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \ . \end{cases}$$

Next we will estimate $|||\partial_{xx}J_{13}(t,x)|||_\alpha$. Writing

$$(2.139)\quad \partial_{xx}J_{13}(t,x) = \int_{\frac{t}{2}}^{t} \partial_x G_{33}(t-\tau,x) * \partial_x[\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}\partial_x v](\tau,x)d\tau$$
$$+ \int_0^{\frac{t}{2}} \partial_{xx}G_{33}(t-\tau,x) * [\{\frac{p_\theta(w+z,\theta)}{e_\theta(w+z,\theta)}(\bar{\theta}+\theta) + d\}\partial_x v](\tau,x)d\tau$$

and using

$$(2.140)\quad \begin{cases} \partial_x G_{33}(t,x) \in C((0,\infty);\ \Lambda_\alpha^{1,\infty}) \\ |||\partial_x G_{33}(t,x)|||_\alpha \leq Mt^{\frac{-1-\alpha}{2}}, \quad \text{for all} \quad t > 0 \ , \end{cases}$$

$$(2.141)\quad \begin{cases} \partial_{xx} G_{33}(t,x) \in C((0,\infty);\ \Lambda_\alpha^{1,\infty}) \\ |||\partial_{xx} G_{33}(t,x)|||_\alpha \leq Mt^{-1-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \ , \end{cases}$$

which follows immediately from (1.95) and a modification of Lemma 1.14, it can be easily deduced that

$$(2.142)\quad \begin{cases} \partial_{xx}J_{13}(t,x) \in C((0,\infty);\ \Lambda_\alpha^{1,\infty}) \\ |||\partial_{xx}J_{13}(t,x)|||_\alpha \leq MK^2 t^{\frac{-1-\alpha}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \ . \end{cases}$$

Lemma 2.17. $J_{14}(t,x) \stackrel{\text{def}}{=} \int_0^t G_{33}(t-\tau,x)*\{\frac{1}{e_\theta(w+z,\theta)}(\partial_x v)^2\}(\tau,x)d\tau$ satisfies the same properties as were stated in Lemma 2.15.

Proof. The assertions concerning $J_{14}(t,x)$ and $\partial_x J_{14}(t,x)$ can be verified analogously to the proof of Lemma 2.10. By the same argument as in Lemma 2.16, we can estimate $\|\partial_{xx}J_{17}(t,x)\|$ and $|||\partial_{xx}J_{17}(t,x)|||_\alpha$. The technical details are left to the reader.

Next we shall present some lemmas which will be used later on.

Lemma 2.18. If $g \in L^1(R)$, then for any $h \in R$,

$$\int_0^h |g(x-t)|dt \leq \int_{-\infty}^{\infty} |g(y+h) - g(y)|dy \tag{2.143}$$

holds for all $x \in R$.

Proof. Let $g_n(x) \in C_0^\infty(R)$, $n = 1,2,\ldots$, such that $g_n \to |g|$ in L^1. Then, we have

$$\begin{aligned}\int_0^h g_n(x-t)dt &= \int_0^h dt \int_{-\infty}^x \partial_y g_n(y-t)dy = \int_{-\infty}^x dy \int_0^h \partial_y g_n(y-t)dt \\ &= \int_{-\infty}^x \{g_n(y) - g_n(y-h)\}dy \leq \int_{-\infty}^{\infty} |g_n(y+h) - g_n(y)|dy .\end{aligned} \tag{2.144}$$

It is obvious that

$$\int_0^h g_n(x-t)dt \to \int_0^h |g(x-t)|dt, \quad \text{for each } x \in R,\ h \in R, \tag{2.145}$$

and

$$\int_{-\infty}^{\infty} |g_n(y+h) - g_n(y)|dy \to \int_{-\infty}^{\infty} ||g(y+h)| - |g(y)||dy, \quad \text{for each } h \in R, \tag{2.146}$$

from which it follows that

$$\int_0^h |g(x-t)|dt \leq \int_{-\infty}^{\infty} ||g(y+h)| - |g(y)||dy \leq \int_{-\infty}^{\infty} |g(y+h) - g(y)|dy, \tag{2.147}$$

for all $x \in R$, $h \in R$.

Lemma 2.19. Let $f_3(x) = f_1(x)f_2(x)$, where $f_1(x) \in L^1 \cap BV$ and $f_2(x) \in \Lambda_\alpha^{1,\infty}$. Then $f_3(x) \in \Lambda_\alpha^{1,\infty}$ and

$$|||f_3(x)|||_\alpha \leq \frac{3}{2}\|\partial_x f_1(x)\|\ |||f_2(x)|||_\alpha . \tag{2.148}$$

Proof. Set $f_{1,n}(x) = f_1(x) * \rho_{\frac{1}{n}}(x)$ and $f_{3,n}(x) = f_{1,n}(x)f_2(x)$. Then, we have

$$\|f_{3,n}(x+h) - f_{3,n}(x)\| < \|f_{1,n}(x+h)\{f_2(x+h) - f_2(x)\}\| \tag{2.149}$$
$$+ \|\{f_{1,n}(x+h) - f_{1,n}(x)\}f_2(x)\| \quad ,$$

and, using Lemma 2.18,

$$\int_{-\infty}^{\infty} dx|f_2(x)| \int_0^h |\partial_x f_{1,n}(x+t)|dt = \int_0^h\int_{-\infty}^{\infty} |f_2(x-t)||\partial_x f_{1,n}(x)|dxdt \tag{2.150}$$
$$< \int_{-\infty}^{\infty} |\partial_x f_{1,n}(x)|dx \int_{-\infty}^{\infty} |f_2(y+h) - f_2(y)|dy$$

for all $h \in R$. Combining these two inequalities, we get

$$|||f_{3,n}(x)|||_\alpha < \{\|f_{1,n}(x)\|_{L^\infty} + \|\partial_x f_{1,n}(x)\|\}|||f_2(x)|||_\alpha \tag{2.151}$$
$$< \frac{3}{2} \|\partial_x f_{1,n}(x)\| \; |||f_2(x)|||_\alpha \quad .$$

Since $f_{1,n}(x) \to f_1(x)$ in L^1, there is a subsequence $\{f_{1,n_k}\}$ such that $f_{1,n_k}(x) \to f_1(x)$ almost everwhere. Moreover,

$$\|f_{1,n}(x)\|_{L^\infty} < \frac{1}{2} \|\partial_x f_{1,n}(x)\| < \frac{1}{2} \|\partial_x f_1(x)\|, \quad \text{for all} \quad n > 1 \quad ,$$

and

$$\|f_1(x)\|_{L^\infty} < \frac{1}{2} \|\partial_x f_1(x)\| \quad .$$

Hence, $f_{3,n_k}(x) \to f_3(x)$ weakly in L^1, which implies $f_{3,n_k}(x+h) \to f_3(x+h)$ weakly in L^1 for each $h \in R$, from which it follows that

$$|||f_3(x)|||_\alpha < \underset{n_k}{\underline{\lim}} \; |||f_{3,n_k}(x)|||_\alpha < \frac{3}{2} \|\partial_x f_1(x)\| \; |||f_2(x)|||_\alpha \quad .$$

Now we proceed to analyze the remaining integrals.

<u>Lemma 2.20</u>. $J_{15}(t,x) \overset{\text{def}}{=} \int_0^t G_{33}(t-\tau,x)*[\{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta](\tau,x)d\tau$ satisfies the same properties as were stated in Lemma 2.15.

<u>Proof</u>. Using Lemma 1.11 and the method of proof of Lemma 2.11, we can easily estimate $\|J_{15}(t,x)\|$ and $\|\partial_x J_{15}(t,x)\|$. For $\partial_{xx}J_{15}(t,x)$, we should employ a different method since $\|\partial_{xx}G_{33}(t-\tau,x)\|$ is not integrable over $(0,t)$. For convenience, let us set

$$(2.152)\qquad B(t,x) = \{\frac{1}{e_\theta(w+z,\theta)} - c\}\partial_{xx}\theta(t,x) \quad .$$

Since $\{\frac{1}{e_\theta(w+z,\theta)} - c\}(t,x) \in C((0,\infty);\ L^1 \cap BV)$ and $\partial_{xx}\theta(t,x) \in C((0,\infty);\ \Lambda_\alpha^{1,\infty})$, we can apply Lemma 2.19 to $B(t,x)$ to obtain

$$(2.153)\qquad \begin{aligned} &B(t,x) \in C((0,\infty);\ \Lambda_\alpha^{1,\infty}) \\ &|||B(t,x)|||_\alpha \le MK^2 t^{\frac{-1-\alpha}{2}}(1+t)^{\frac{-1-\alpha}{2}} \ , \quad \text{for all} \quad t > 0 \quad . \end{aligned}$$

Next define

$$(2.154)\qquad \Gamma_\varepsilon(t,x) = \int_0^{\max(t-\varepsilon,0)} G_{33}(t-\tau,x)*B(\tau,x)d\tau \quad .$$

Then, obviously $\Gamma_\varepsilon(t,x) \to J_{15}(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$ as $\varepsilon \to 0$, which implies $\partial_{xx}\Gamma_\varepsilon(t,x) \to \partial_{xx}J_{15}(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$. Noticing that, for each $\varepsilon > 0$,

$$(2.155)\qquad \begin{aligned} \|\partial_{xx}\Gamma_\varepsilon(t_1,x) - \partial_{xx}\Gamma_\varepsilon(t_2,x)\| &\le \int_{\max(t_2-\varepsilon,0)}^{\max(t_1-\varepsilon,0)} \|\partial_{xx}G_{33}(t_1-\tau,x)\|\ \|B(\tau,x)\|d\tau \\ &+ \int_0^{\max(t_2-\varepsilon,0)} \|\partial_{xx}G_{33}(t_1-\tau,x) - \partial_{xx}G_{33}(t_2-\tau,x)\|\ \|B(\tau,x)\|d\tau \end{aligned}$$

holds for all $0 \le t_2 < t_1$, we conclude that

$$(2.156)\qquad \partial_{xx}\Gamma_\varepsilon(t,x) \in C([0,\infty);\ L^1) \quad .$$

In the mean time, for $0 < \varepsilon < t$,

$$(2.157)\qquad \begin{aligned} \partial_{xx}\Gamma_\varepsilon(t,x) &= \int_0^{t-\varepsilon}\int_{-\infty}^{\infty} \partial_{xx}G_{33}(t-\tau,x-y)B(\tau,y)dyd\tau \\ &= \int_0^{t-\varepsilon}\int_{-\infty}^{\infty} \partial_{xx}G_{33}(t-\tau,x-y)\{B(\tau,y) - B(\tau,x)\}dyd\tau \end{aligned}$$

is valid from Lemma 1.11. Now fix any closed interval $[T_1,T_2] \subset (0,\infty)$. Then, using (2.157), we find that

$$(2.158) \quad \|\partial_{xx}\Gamma_{\varepsilon_1}(t,x) - \partial_{xx}\Gamma_{\varepsilon_2}(t,x)\|$$

$$< \int_{t-\varepsilon_2}^{t-\varepsilon_1} d\tau \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |x-y|^{\alpha} |\partial_{xx}G_{33}(t-\tau,x-y)| \frac{|B(\tau,y)-B(\tau,x)|}{|y-x|^{\alpha}}$$

$$= \int_{t-\varepsilon_2}^{t-\varepsilon_1} d\tau \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} drdq |r|^{\alpha} |\partial_{xx}G_{33}(t-\tau,r)| \frac{|B(\tau,q)-B(\tau,q+r)|}{|r|^{\alpha}}$$

$$< \int_{t-\varepsilon_2}^{t-\varepsilon_1} d\tau \; |||B(\tau,x)|||_{\alpha} \int_{-\infty}^{\infty} dr \; |r|^{\alpha} |\partial_{xx}G_{33}(t-\tau,r)|$$

$$< MK^2 \int_{t-\varepsilon_2}^{t-\varepsilon_1} d\tau \; \tau^{\frac{-1-\alpha}{2}} (1+\tau)^{\frac{-1-\alpha}{2}} \{(t-\tau)^{-1+\frac{\alpha}{2}} + (t-\tau)^{-1+\alpha}\},$$

by (2.153) and (1.97) ,

$$< MK^2 T_1^{\frac{-1-\alpha}{2}} (2+T_1)^{\frac{-1-\alpha}{2}} \{\varepsilon_1^{\frac{\alpha}{2}} + \varepsilon_2^{\frac{\alpha}{2}} + \varepsilon_1^{\alpha} + \varepsilon_2^{\alpha}\}$$

holds for all $0 < \varepsilon_1 < \varepsilon_2 < \frac{1}{2} T_1$ and all $t \in [T_1,T_2]$. Hence, $\partial_{xx}\Gamma_{\varepsilon}(t,x)$ converges in L^1 uniformly on each compact subset of $(0,\infty)$ as $\varepsilon \to 0$, which implies

$$(2.159) \qquad \partial_{xx}J_{15}(t,x) \in C((0,\infty);\, L^1)$$

and

$$(2.160) \qquad \begin{aligned} \partial_{xx}J_{15}(t,x) &= \lim_{\varepsilon \to 0} \int_0^{\max(t-\varepsilon,0)} \int_{-\infty}^{\infty} \partial_{xx}G_{33}(t-\tau,x-y)\{B(\tau,y) - B(\tau,x)\}dyd\tau \\ &= \int_0^t \int_{-\infty}^{\infty} \partial_{xx}G_{33}(t-\tau,x-y)\{B(\tau,y) - B(\tau,x)\}dyd\tau \end{aligned} ,$$

for each $t > 0$. Using this formula and the fact that $0 < \alpha < \frac{1}{3}$, we can estimate $\|\partial_{xx}J_{15}(t,x)\|$ in parallel with (2.158):

$$(2.161) \quad \|\partial_{xx}J_{15}(t,x)\| < \int_0^t d\tau \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; |x-y|^{\alpha} |\partial_{xx}G_{33}(t-\tau,x-y)| \frac{|B(\tau,y)-B(\tau,x)|}{|y-x|^{\alpha}}$$

$$< MK^2 \int_0^t d\tau \; \tau^{\frac{-1-\alpha}{2}} (1+\tau)^{\frac{-1-\alpha}{2}} \{(t-\tau)^{-1+\frac{\alpha}{2}} + (t-\tau)^{-1+\alpha}\}$$

$$\leq MK^2 t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad .$$

Next we shall estimate $|||\partial_{xx}J_{15}(t,x)|||_\alpha$ for each $t > 0$, and prove that $\partial_{xx}J_{15}(t,x) \in C((0,\infty);\ \Lambda_\alpha^{1,\infty})$. Fix any $t > 0$. If $\sqrt{\frac{t}{2}} \leq |h|$, then

$$(2.162) \qquad \frac{\|\partial_{xx}J_{15}(t,x+h)-\partial_{xx}J_{15}(t,x)\|}{|h|^\alpha} \leq \frac{2^{1+\frac{\alpha}{2}}}{t^{\frac{\alpha}{2}}} \|\partial_{xx}J_{15}(t,x)\| \leq MK^2 t^{\frac{-1-\alpha}{2}}(1+t)^{-\frac{\alpha}{2}} \quad .$$

Now suppose $|h| < \sqrt{\frac{t}{2}}$. $\partial_{xx}J_{15}(t,x)$ can be written in the form

$$(2.163) \qquad \begin{aligned} \partial_{xx}J_{15}(t,x) &= \int_{t-\eta}^{t}\int_{-\infty}^{\infty} \partial_{xx}G_{33}(t-\tau,x-y)\{B(\tau,y) - B(\tau,x)\}dyd\tau \\ &+ \int_0^{t-\eta}\int_{-\infty}^{\infty} \partial_{xx}G_{33}(t-\tau,x-y)B(\tau,y)dyd\tau, \quad \text{for any} \quad 0 < \eta < t \quad . \end{aligned}$$

Let us denote the first double integral on the right-hand side by $I_1(t,x)$ and the second one by $I_2(t,x)$. Then,

(2.164)

$$\frac{1}{|h|^\alpha}\|\partial_{xx}J_{15}(t,x+h) - \partial_{xx}J_{15}(t,x)\| \leq \frac{2}{|h|^\alpha}\|I_1(t,x)\| + \frac{1}{|h|^\alpha}\| I_2(t,x+h)-I_2(t,x)\| \quad .$$

By taking $\eta = h^2 < \frac{t}{2}$, we have

$$(2.165) \qquad \begin{aligned} \frac{1}{|h|^\alpha}\|I_1(t,x)\| &\leq \frac{1}{|h|^\alpha}\int_{t-\eta}^{t}d\tau \int_{-\infty}^{\infty}dx \int_{-\infty}^{\infty}dy|x-y|^\alpha|\partial_{xx}G_{33}(t-\tau,x-y)|\frac{|B(\tau,y)-B(\tau,x)|}{|y-x|^\alpha} \\ &\leq \frac{MK^2}{|h|^\alpha}\int_{t-\eta}^{t}d\tau\{(t-\tau)^{-1+\frac{\alpha}{2}} + (t-\tau)^{-1+\alpha}\}\tau^{\frac{-1-\alpha}{2}}(1+\tau)^{\frac{-1-\alpha}{2}} \\ &\leq MK^2 t^{\frac{-1-\alpha}{2}}(1+t)^{-\frac{\alpha}{2}} \quad . \end{aligned}$$

By virtue of the identity

$$(2.166) \qquad \begin{aligned} &\int_0^{t-\eta}\int_{-\infty}^{\infty}\{\partial_{xx}G_{33}(t-\tau,h+x-y) - \partial_{xx}G_{33}(t-\tau,x-y)\}B(\tau,y)dyd\tau \\ &= \int_0^{t-\eta}\int_{-\infty}^{\infty}\{\int_0^h \partial_{xxx}G_{33}(t-\tau,x-y+\zeta)d\zeta\}\{B(\tau,y) - B(\tau,x)\}dyd\tau \quad , \end{aligned}$$

which follows from Lemma 1.11, we find that

$$(2.167)\quad \frac{1}{|h|^\alpha}\| I_2(t,x+h) - I_2(t,x)\|$$

$$\le \frac{1}{|h|^\alpha}\int_0^{t-\eta}d\tau \int_{-\infty}^{\infty}dx \int_{-\infty}^{\infty}dy\{\int_0^{|h|}|x-y|^\alpha|\partial_{xxx}G_{33}(t-\tau,x-y+\zeta)|d\zeta\}\frac{|B(\tau,y)-B(\tau,x)|}{|y-x|^\alpha}$$

Substituting $q = y$, $r = x-y$ and using the inequality

$$(2.168)\qquad |r|^\alpha \le 2^\alpha|r+\zeta|^\alpha + 2^\alpha|\zeta|^\alpha, \quad \text{for all } r,\zeta \in R ,$$

(2.167) becomes

$$\frac{1}{|h|^\alpha}\| I_2(t,x+h) - I_2(t,x)\|$$

$$\le \frac{2^\alpha}{|h|^\alpha}\int_0^{t-\eta}d\tau \int_{-\infty}^{\infty}dr \int_{-\infty}^{\infty}dq\{\int_0^{|h|}|\partial_{xxx}G_{33}(t-\tau,r+\zeta)|(|r+\zeta|^\alpha + |\zeta|^\alpha)$$

$$(2.169)\qquad d\zeta\}\frac{|B(\tau,q+r)-B(\tau,q)|}{|r|^\alpha}$$

$$\le 2^\alpha|h|^{1-\alpha}\int_0^{t-\eta}d\tau|||B(\tau,x)|||_\alpha \int_{-\infty}^{\infty}dr\ |r|^\alpha|\partial_{xxx}G_{33}(t-\tau,r)|$$

$$+ \frac{2^\alpha}{1+\alpha}|h|\int_0^{t-\eta}d\tau\ |||B(\tau,x)|||_\alpha \int_{-\infty}^{\infty}dr\ |\partial_{xxx}G_{33}(t-\tau,r)|$$

$$\le MK^2|h|^{1-\alpha}\int_0^{t-\eta}d\tau\ \tau^{\frac{-1-\alpha}{2}}(1+\tau)^{\frac{-1-\alpha}{2}}\{(t-\tau)^{\frac{-3+\alpha}{2}} + (t-\tau)^{-\frac{3}{2}+\alpha}\}$$

$$+ MK^2|h|\int_0^{t-\eta}d\tau\ \tau^{\frac{-1-\alpha}{2}}(1+\tau)^{\frac{-1-\alpha}{2}}(t-\tau)^{-\frac{3}{2}} .$$

Taking $\eta = h^2 < \frac{t}{2}$ as before and breaking each integral of the last two terms into two parts by $\int_0^{t-\eta} = \int_{\frac{t}{2}}^{t-\eta} + \int_0^{\frac{t}{2}}$, we can obtain the estimate:

$$(2.170)\quad \frac{1}{|h|^\alpha}\| I_2(t,x+h) - I_2(t,x)\| \le MK^2 t^{\frac{-1-\alpha}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } 0 < |h| < \sqrt{\frac{t}{2}} .$$

Combining (2.162), (2.165) and (2.170), we conclude that

$$\text{(2.171)} \qquad |||\partial_{xx}J_{15}(t,x)|||_\alpha \leq MK^2 t^{\frac{-1-\alpha}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 .$$

Finally, we shall prove the continuity in $t > 0$. Fix any t_1, t_2 such that $0 < t_1 - t_2 \leq \frac{1}{4} t_1$ and $0 < \varepsilon \leq t_2 < t_1 \leq L$. By (2.160), (1.96), we can write, provided $0 < \eta \leq \frac{t_2}{2}$,

$$\text{(2.172)} \qquad \partial_{xx}J_{15}(t_1,x) - \partial_{xx}J_{15}(t_2,x) = \int_0^\eta\int_{-\infty}^\infty \partial_{xx}G_{33}(\tau,x-y)\{B(t_1-\tau,y) - B(t_2-\tau,y)$$

$$- B(t_1-\tau,x) + B(t_2-\tau,x)\}dyd\tau$$

$$+ \int_{t_2}^{t_1}\int_{-\infty}^\infty \partial_{xx}G_{33}(\tau,x-y)B(t_1-\tau,y)dyd\tau$$

$$+ \int_{\frac{t_2}{2}}^{t_2}\int_{-\infty}^\infty \partial_{xx}G_{33}(\tau,x-y)\{B(t_1-\tau,y) - B(t_2-\tau,y)\}dyd\tau$$

$$+ \int_\eta^{\frac{t_2}{2}}\int_{-\infty}^\infty \partial_{xx}G_{33}(\tau,x-y)\{B(t_1-\tau,y) - B(t_2-\tau,y)\}dyd\tau .$$

Denote the integrals on the right-hand side by $E_1(t_1,t_2,x)$, $E_2(t_1,t_2,x)$, $E_3(t_1,t_2,x)$ and $E_4(t_1,t_2,x)$ according to their orders. Analogously to (2.165), (2.169), it holds that

$$\text{(2.173)} \qquad \frac{1}{|h|^\alpha}\|E_1(t_1,t_2,x+h) - E_1(t_1,t_2,x)\| \leq \frac{2}{|h|^\alpha}\|E_1(t_1,t_2,x)\|$$

$$\leq \frac{2}{|h|^\alpha}\int_0^\eta d\tau \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \; |x-y|^\alpha|\partial_{xx}G_{33}(\tau,x-y)|$$

$$\frac{|B(t_1-\tau,y)-B(t_2-\tau,y)-B(t_1-\tau,x)+B(t_2-\tau,x)|}{|y-x|^\alpha}$$

$$\leq \frac{M}{|h|^\alpha}\int_0^\eta d\tau\, \{\tau^{-1+\frac{\alpha}{2}} + \tau^{-1+\alpha}\}|||B(t_1-\tau,x) - B(t_2-\tau,x)|||_\alpha$$

$$\leq M(1+t_2^{\frac{\alpha}{2}}) \sup_{0\leq\tau\leq\frac{t_2}{2}} |||B(t_1-\tau,x) - B(t_2-\tau,x)|||_\alpha, \quad \text{provided } \eta = h^2 \leq \frac{t_2}{2},$$

(2.174)
$$\frac{1}{|h|^\alpha} \| E_2(t_1,t_2,x+h) - E_2(t_1,t_2,x)\|$$
$$\leq MK^2 \int_{t_2}^{t_1} d\tau\, |h|^{1-\alpha}\{\tau^{\frac{-3+\alpha}{2}} + \tau^{-\frac{3}{2}+\alpha}\}(t_1-\tau)^{\frac{-1-\alpha}{2}}(1+t_1-\tau)^{\frac{-1-\alpha}{2}}$$
$$+ MK^2 \int_{t_2}^{t_1} d\tau\, |h|\, \tau^{-\frac{3}{2}}(t_1-\tau)^{\frac{-1-\alpha}{2}}(1+t_1-\tau)^{\frac{-1-\alpha}{2}}$$
$$\leq MK^2(t_1-t_2)^{\frac{1-\alpha}{2}}\{|h|^{1-\alpha}(t_2^{\frac{-3+\alpha}{2}} + t_2^{-\frac{3}{2}+\alpha}) + |h|t_2^{-\frac{3}{2}}\} .$$

For $E_3(t_1,t_2,x)$, we need to use the expression:

(2.175)
$$E_3(t_1,t_2,x) = -\int_0^{t_1-t_2}\int_{-\infty}^{\infty} \partial_{xx}G_{33}(t_2-\tau,x-y)B(\tau,y)dyd\tau$$
$$+ \int_{t_1-t_2}^{\frac{t_2}{2}}\int_{-\infty}^{\infty}\{\partial_{xx}G_{33}(t_1-\tau,x-y) - \partial_{xx}G_{33}(t_2-\tau,x-y)\}B(\tau,y)dyd\tau$$
$$+ \int_{\frac{t_2}{2}}^{\frac{t_2}{2}+(t_1-t_2)}\int_{-\infty}^{\infty} \partial_{xx}G_{33}(t_1-\tau,x-y)B(\tau,y)dyd\tau .$$

Denote the integrals on the right-hand side by $E_5(t_1,t_2,x)$, $E_6(t_1,t_2,x)$ and $E_7(t_1,t_2,x)$ according to their orders. Then, imitating the development of (2.169), we have

(2.176)
$$\frac{1}{|h|^\alpha} \| E_5(t_1,t_2,x+h) - E_5(t_1,t_2,x)\|$$

$$\leq MK^2 \int_0^{t_1-t_2} d\tau \; |h|^{1-\alpha}\{(t_2-\tau)^{\frac{-3+\alpha}{2}} + (t_2-\tau)^{-\frac{3}{2}+\alpha}\}\tau^{\frac{-1-\alpha}{2}}(1+\tau)^{\frac{-1-\alpha}{2}}$$

$$+ MK^2 \int_0^{t_1-t_2} d\tau \; |h|(t_2-\tau)^{-\frac{3}{2}}\tau^{\frac{-1-\alpha}{2}}(1+\tau)^{\frac{-1-\alpha}{2}}$$

$$\leq MK^2(t_1-t_2)^{\frac{1-\alpha}{2}}\{|h|^{1-\alpha}(t_2^{\frac{-3+\alpha}{2}} + t_2^{-\frac{3}{2}+\alpha}) + |h|t_2^{-\frac{3}{2}}\} \;,$$

(2.177)
$$\frac{1}{|h|^\alpha}\|E_6(t_1,t_2,x+h) - E_6(t_1,t_2,x)\|$$

$$\leq MK^2|h|^{1-\alpha}\int_{t_1-t_2}^{\frac{t_2}{2}} d\tau \; \tau^{\frac{-1-\alpha}{2}}(1+\tau)^{\frac{-1-\alpha}{2}} \sup_{\lambda\in[t_1-t_2,\frac{t_2}{2}]}$$

$$\int_{-\infty}^{\infty} dr \; |r|^\alpha|\partial_{xxx}G_{33}(t_1-\lambda,r) - \partial_{xxx}G_{33}(t_2-\lambda,r)|$$

$$+ MK^2 \; |h| \int_{t_1-t_2}^{\frac{t_2}{2}} d\tau \; \tau^{\frac{-1-\alpha}{2}}(1+\tau)^{\frac{-1-\alpha}{2}}$$

$$\sup_{\lambda\in[t_1-t_2,\frac{t_2}{2}]} \int_{-\infty}^{\infty} dr \; |\partial_{xxx}G_{33}(t_1-\lambda,r) - \partial_{xxx}G_{33}(t_2-\lambda,r)|$$

$$\leq MK^2\{|h|^{1-\alpha}t_2^{\frac{1-\alpha}{2}} + |h|t_2^{\frac{1-\alpha}{2}}\} \sup_{\lambda\in[t_1-t_2,\frac{t_2}{2}]}$$

$$\int_{-\infty}^{\infty} dr(1+|r|^\alpha)|\partial_{xxx}G_{33}(t_1-\lambda,r) - \partial_{xxx}G_{33}(t_2-\lambda,r)|$$

(2.178)
$$\frac{1}{|h|^\alpha}\|E_7(t_1,t_2,x+h) - E_7(t_1,t_2,x)\|$$

$$\leq MK^2 \; |h|^{1-\alpha}\int_{\frac{t_2}{2}}^{\frac{t_2}{2}+(t_1-t_2)} d\tau\{(t_1-\tau)^{\frac{-3+\alpha}{2}} + (t_1-\tau)^{-\frac{3}{2}+\alpha}\}$$

$$\times \tau^{\frac{-1-\alpha}{2}}(1+\tau)^{\frac{-1-\alpha}{2}}$$

$$+ MK^2 |h| \int_{\frac{t_2}{2}}^{\frac{t_2}{2}+(t_1-t_2)} d\tau (t_1-\tau)^{-\frac{3}{2}} \tau^{\frac{-1-\alpha}{2}} (1+\tau)^{\frac{-1-\alpha}{2}}$$

$$\leq MK^2(|h|^{1-\alpha} + |h|) t_2^{\frac{-1-\alpha}{2}} (1+t_2)^{\frac{-1-\alpha}{2}} \{-(\frac{t_2}{2}+t_1-t_2)^{\frac{-1+\alpha}{2}} + (\frac{t_2}{2})^{\frac{-1+\alpha}{2}}$$

$$- (\frac{t_2}{2}+t_1-t_2)^{-\frac{1}{2}+\alpha} + (\frac{t_2}{2})^{-\frac{1}{2}+\alpha} - (\frac{t_2}{2}+t_1-t_2)^{-\frac{1}{2}} + (\frac{t_2}{2})^{-\frac{1}{2}}\} \quad .$$

Repeating the previous argument, we obtain

$$\frac{1}{|h|^{\alpha}} \| E_4(t_1,t_2,x+h) - E_4(t_1,t_2,x) \|$$

(2.179)

$$\leq MK^2 \int_{\eta}^{\frac{t_2}{2}} d\tau |h|^{1-\alpha} \{\tau^{\frac{-3+\alpha}{2}} + \tau^{-\frac{3}{2}+\alpha}\} \sup_{\lambda \in [\eta, \frac{t_2}{2}]} |||B(t_1-\lambda,x) - B(t_2-\lambda,x)|||_{\alpha}$$

$$+ MK^2 \int_{\eta}^{\frac{t_2}{2}} d\tau \; |h| \tau^{-\frac{3}{2}} \sup_{\lambda \in [\eta, \frac{t_2}{2}]} |||B(t_1-\lambda,x) - B(t_2-\lambda,x)|||_{\alpha}$$

$$\leq MK^2 \{|h|^{1-\alpha} t_2^{\frac{-1+\alpha}{2}} + |h|^{1-\alpha} \eta^{\frac{-1+\alpha}{2}} + |h|^{1-\alpha} t_2^{-\frac{1}{2}+\alpha} + |h|^{1-\alpha} \eta^{-\frac{1}{2}+\alpha}$$

$$+ |h| t_2^{-\frac{1}{2}} + |h| \eta^{-\frac{1}{2}}\} \times \sup_{\lambda \in [0, \frac{t_2}{2}]} |||B(t_1-\lambda,x) - B(t_2-\lambda,x)|||_{\alpha}$$

$$\leq MK^2 (1 + t_2^{\frac{\alpha}{2}}) \sup_{\lambda \in [0, \frac{t_2}{2}]} |||B(t_1-\lambda,x) - B(t_2-\lambda,x)|||_{\alpha},$$

provided $\eta = h^2 < \frac{t_2}{2}$.

By (2.173), (2.174), (2.176) to (2.179), we conclude that

(2.180)

$$\lim_{\substack{|t_1-t_2| \to 0 \\ \varepsilon < t_2 < t_1 < L}} \sup_{0<|h|<\frac{\varepsilon}{2}} \frac{1}{|h|^{\alpha}} \| \partial_{xx} J_{15}(t_1,x+h) - \partial_{xx} J_{15}(t_2,x+h) - \partial_{xx} J_{15}(t_1,x) + \partial_{xx} J_{15}(t_2,x) \| = 0 \quad .$$

On the other hand

$$(2.181) \qquad \lim_{\substack{|t_1-t_2|\to 0 \\ \varepsilon<t_2<t_1<L}} \sup_{\frac{\varepsilon}{2}<|h|} \frac{1}{|h|^\alpha} \|\partial_{xx}J_{15}(t_1,x+h) - \partial_{xx}J_{15}(t_2,x+h) - \partial_{xx}J_{15}(t_1,x) + \partial_{xx}J_{15}(t_2,x)\|$$

$$< 2\left(\frac{2}{\varepsilon}\right)^\alpha \lim_{\substack{|t_1-t_2|\to 0 \\ \varepsilon<t_2<t_1<L}} \|\partial_{xx}J_{15}(t_1,x) - \partial_{xx}J_{15}(t_2,x)\| = 0 \quad .$$

Since ε, L were chosen arbitrarily, (2.180) and (2.181) yield

$\partial_{xx}J_{15}(t,x) \in C((0,\infty);\ \Lambda_\alpha^{1,\infty})$.

Now let us summarize what we have obtained in Theorem 1.13 and Lemmas 2.3 to 2.20.

<u>Proposition 2.21</u>. Suppose $\tilde{w}(t,x)$, $\tilde{z}(t,x)$, $\tilde{v}(t,x)$ and $\tilde{\theta}(t,x)$ are defined by (2.26) to (2.29). Then we have:

(I) $\tilde{w}(t,x) \in C([0,\infty);\ L^1)$, $\tilde{w}(0,x) = u_0(x)$, $\partial_x\tilde{w}(t,x) \in C([0,\infty);\ M)$

$\partial_t\tilde{w}(t,x) \in C((0,\infty);\ L^1)$, $\partial_t\partial_x\tilde{w}(t,x) \in C((0,\infty);\ M)$

$$(2.182) \qquad \|\tilde{w}(t,x)\| < (\mu M_1 + M_2K^2)(1+t)^{\frac{1-m}{2}}, \quad \text{for all} \quad t > 0 \quad ,$$

$$(2.183) \qquad \|\partial_x\tilde{w}(t,x)\| < (\mu M_1 + M_2K^2)(1+t)^{-\frac{m}{2}}, \quad \text{for all} \quad t > 0 \quad ,$$

$$(2.184) \qquad \|\partial_t\tilde{w}(t,x)\| < (\mu M_1 + M_2K^2)(1+t)^{-\frac{m}{2}}, \quad \text{for all} \quad t > 0 \quad ,$$

$$(2.185) \quad \|\partial_t\partial_x\tilde{w}(t,x)\| < (\mu M_1 + M_2K^2)(t^{-\frac{1}{2}} + t^{-\frac{\alpha}{2}})(1+t)^{-\frac{m}{2}}, \quad \text{for all} \quad t > 0 \quad ,$$

where μ is the bound for the size of initial data (see Theorem 1.13) and M_1, M_2 are constants independent of μ, K and t.

(II) $\tilde{z}(t,x) \in C([0,\infty);\ L^1)$, $\tilde{z}(0,x) = 0$, $\partial_x\tilde{z}(t,x) \in C([0,\infty);\ L^1)$

$\partial_{xx}\tilde{z}(t,x) \in C((0,\infty);\ M)$

$$(2.186)\qquad \|\tilde{z}(t,x)\| \leq \mu M_1 + M_2K^2, \quad \text{for all} \quad t \geq 0 \quad ,$$

$$(2.187)\qquad \|\partial_x\tilde{z}(t,x)\| \leq (\mu M_1 + M_2K^2)(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t \geq 0 \quad ,$$

$$(2.188)\qquad \|\partial_{xx}\tilde{z}(t,x)\| \leq (\mu M_1 + M_2K^2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad .$$

(III) $\tilde{v}(t,x) \in C([0,\infty); L^1)$, $\tilde{v}(0,x) = v_0(x)$, $\partial_x\tilde{v}(t,x) \in C((0,\infty); L^1)$,

$\partial_{xx}\tilde{v}(t,x) \in C((0,\infty); M)$

$$(2.189)\qquad \|\tilde{v}(t,x)\| \leq \mu M_1 + M_2K^2, \quad \text{for all} \quad t \geq 0 \quad ,$$

$$(2.190)\qquad \|\partial_x\tilde{v}(t,x)\| \leq (\mu M_1 + M_2K^2)(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad ,$$

$$(2.191)\qquad \|\partial_{xx}\tilde{v}(t,x)\| \leq (\mu M_1 + M_2K^2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad .$$

(IV) $\tilde{\theta}(t,x) \in C([0,\infty); L^1)$, $\tilde{\theta}(0,x) = \theta_0(x)$, $\partial_x\tilde{\theta}(t,x) \in C((0,\infty); L^1)$

$\partial_{xx}\tilde{\theta}(t,x) \in C((0,\infty); \Lambda_\alpha^{1,\infty})$

$$(2.192)\qquad \|\tilde{\theta}(t,x)\| \leq \mu M_1 + M_2K^2, \quad \text{for all} \quad t \geq 0 \quad ,$$

$$(2.193)\qquad \|\partial_x\tilde{\theta}(t,x)\| \leq (\mu M_1 + M_2K^2)(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad ,$$

$$(2.194)\qquad \|\partial_{xx}\tilde{\theta}(t,x)\| \leq (\mu M_1 + M_2K^2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad ,$$

$$(2.195)\qquad |||\partial_{xx}\tilde{\theta}(t,x)|||_\alpha \leq (\mu M_1 + M_2K^2)t^{\frac{-1-\alpha}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad .$$

From this proposition and Equations (2.30), we derive

Proposition 2.22. It holds that

$$(2.196)\qquad \partial_t\tilde{w}(t,x) + \partial_t\tilde{z}(t,x) = \partial_x\tilde{v}(t,x) \quad \text{in} \quad \mathcal{D}^*((0,\infty) \times R) \quad ,$$

$$(2.197)\qquad \begin{cases} \partial_t \tilde{v}(t,x) \in C((0,\infty);\ M) \\ \|\partial_t \tilde{v}(t,x)\| \le M_3(\mu M_1 + M_2K^2)t^{-\frac{1}{2}}, \text{ for all } t > 0 \end{cases},$$

$$(2.198)\qquad \begin{cases} \partial_t \tilde{\theta}(t,x) \in C((0,\infty);\ L^1) \\ \|\partial_t \tilde{\theta}(t,x)\| \le M_3(\mu M_1 + M_2K^2)t^{-\frac{1}{2}}, \text{ for all } t > 0 \end{cases},$$

$$(2.199)\quad \|\partial_t \tilde{\theta}(t,x) - d\partial_x \tilde{v}(t,x)\| \le M_3(\mu M_1 + M_2K^2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \text{ for all } t > 0 ,$$

where M_3 is a constant independent of μ, M_1, M_2, K and t.

These two propositions complete our proof that $(\tilde{w}(t,x), \tilde{z}(t,x), \tilde{v}(t,x), \tilde{\theta}(t,x)) \in X$, provided that

$$(2.200)\qquad \mu(1 + M_3)(1+M_1) \le \frac{1}{2} K ,$$

$$(2.201)\qquad (1 + M_3)M_2K \le \frac{1}{2} .$$

(Step III). We shall prove that T is a contraction. Let $(\tilde{w}_i,\tilde{z}_i,\tilde{v}_i,\tilde{\theta}_i) = T(w_i,z_i,v_i,\theta_i)$, for $(w_i,z_i,v_i,\theta_i) \in X$, $i = 1,2$. Then, we need the following expressions:

$$(2.202)\quad \tilde{w}_1(t,x) - \tilde{w}_2(t,x) = -\int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x) * \{\sigma_1(\tau,x) - \sigma_2(\tau,x)\}d\tau$$

$$+\int_0^{\frac{t}{2}} e^{-a(t-\tau)}[\{p(w_1+z_1,\theta_1) + aw_1 + az_1 + b\theta_1\}$$

$$- \{p(w_2+z_2,\theta_2) + aw_2 + az_2 + b\theta_2\}](\tau,x)d\tau ,$$

where $\sigma_i(t,x) = p(w_i+z_i,\theta_i)_x - p_u(w_i+z_i,\theta_i)\partial_x z_i - p_\theta(w_i+z_i,\theta_i)\partial_x\theta_i + a\partial_x w_i$, $i = 1,2$.

$$(2.203)\quad \tilde{z}_1(t,x) - \tilde{z}_2(t,x) = -\int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*$$

$$[\{p_u(w_1+z_1,\theta_1)+a\}\partial_x z_1 - \{p_u(w_2+z_2,\theta_2)+a\}\partial_x z_2$$

$$+ \{p_\theta(w_1+z_1,\theta_1)+b\}\partial_x\theta_1 - \{p_\theta(w_2+z_2,\theta_2)+b\}\partial_x\theta_2](\tau,x)d\tau$$

$$- \int_0^{\frac{t}{2}} H_5(t-\tau,x)*[\{p(w_1+z_1,\theta_1) + aw_1 + az_1 + b\theta_1\}$$

$$- \{p(w_2+z_2,\theta_2) + aw_2 + az_2 + b\theta_2\}](\tau,x)d\tau$$

$$- \int_0^t G_{13}(t-\tau,x)*[\{\frac{p_\theta(w_1+z_1,\theta_1)}{e_\theta(w_1+z_1,\theta_1)}(\bar{\theta}+\theta_1) + d\}\partial_x v_1$$

$$- \{\frac{p_\theta(w_2+z_2,\theta_2)}{e_\theta(w_2+z_2,\theta_2)}(\bar{\theta}+\theta_2) + d\}\partial_x v_2](\tau,x)d\tau$$

$$+ \int_0^t G_{13}(t-\tau,x)*[\frac{1}{e_\theta(w_1+z_1,\theta_1)}(\partial_x v_1)^2 - \frac{1}{e_\theta(w_2+z_2,\theta_2)}(\partial_x v_2)^2](\tau,x)d\tau$$

$$+ \int_0^t G_{13}(t-\tau,x)*[\{\frac{1}{e_\theta(w_1+z_1,\theta_1)} - c\}\partial_{xx}\theta_1 - \{\frac{1}{e_\theta(w_2+z_2,\theta_2)} - c\}\partial_{xx}\theta_2](\tau,x)d\tau$$

$$(2.204)\quad \tilde{v}_1(t,x) - \tilde{v}_2(t,x) = -\int_0^t G_{22}(t-\tau,x)*[\partial_x\{p(w_1+z_1,\theta_1) + aw_1 + az_1 + b\theta_1\}$$

$$- \partial_x\{p(w_2+z_2,\theta_2) + aw_2 + az_2 + b\theta_2\}](\tau,x)d\tau$$

$$- \int_0^t G_{23}(t-\tau,x)*[\{\frac{p_\theta(w_1+z_1,\theta_1)}{e_\theta(w_1+z_1,\theta_1)}(\bar{\theta}+\theta_1) + d\}\partial_x v_1$$

$$- \{\frac{p_\theta(w_2+z_2,\theta_2)}{e_\theta(w_2+z_2,\theta_2)}(\bar{\theta}+\theta_2) + d\}\partial_x v_2](\tau,x)d\tau$$

$$+ \int_0^t G_{23}(t-\tau,x)*\{\frac{1}{e_\theta(w_1+z_1,\theta_1)}(\partial_x v_1)^2 - \frac{1}{e_\theta(w_2+z_2,\theta_2)}(\partial_x v_2)^2\}(\tau,x)d\tau$$

$$+ \int_0^t G_{23}(t-\tau,x)*[\{\frac{1}{e_\theta(w_1+z_1,\theta_1)} - c\}\partial_{xx}\theta_1 - \{\frac{1}{e_\theta(w_2+z_2,\theta_2)} - c\}\partial_{xx}\theta_2](\tau,x)d\tau$$

$$(2.205)\ \tilde{\theta}_1(t,x) - \tilde{\theta}_2(t,x) = -\int_0^t G_{32}(t-\tau,x)*[\partial_x\{p(w_1+z_1,\theta_1) + aw_1 + az_1 + b\theta_1\}$$

$$- \partial_x\{p(w_2+z_2,\theta_2) + aw_2 + az_2 + b\theta_2\}](\tau,x)d\tau$$

$$- \int_0^t G_{33}(t-\tau,x)*[\{\frac{p_\theta(w_1+z_1,\theta_1)}{e_\theta(w_1+z_1,\theta_1)}(\bar{\theta}+\theta_1) + d\}\partial_x v_1$$

$$- \{\frac{p_\theta(w_2+z_2,\theta_2)}{e_\theta(w_2+z_2,\theta_2)}(\bar{\theta}+\theta_2) + d\}\partial_x v_2](\tau,x)d\tau$$

$$+ \int_0^t G_{33}(t-\tau,x)*\{\frac{1}{e_\theta(w_1+z_1,\theta_1)}(\partial_x v_1)^2 - \frac{1}{e_\theta(w_2+z_2,\theta_2)}(\partial_x v_2)^2\}(\tau,x)d\tau$$

$$+ \int_0^t G_{33}(t-\tau,x)*[\{\frac{1}{e_\theta(w_1+z_1,\theta_1)} - c\}\partial_{xx}\theta_1 - \{\frac{1}{e_\theta(w_2+z_2,\theta_2)} - c\}\partial_{xx}\theta_2](\tau,x)d\tau\ .$$

For convenience, let Φ_i denote (w_i,z_i,v_i,θ_i), $i = 1,2,$ and recall that the metric $d(\cdot,\cdot)$ was defined by (2.18). For technical details of proofs of the following lemmas, the reader should go back to the proofs in (Step II).

Lemma 2.23. It holds that

$$(2.206)\qquad \|\tilde{w}_1(t,x) - \tilde{w}_2(t,x)\| \le MKd(\Phi_1,\Phi_2)(1+t)^{\frac{1-m}{2}},\ \text{for all}\ t \ge 0\ ,$$

$$(2.207)\qquad \|\partial_x\tilde{w}_1(t,x) - \partial_x\tilde{w}_2(t,x)\| \le MKd(\Phi_1,\Phi_2)(1+t)^{-\frac{m}{2}},\ \text{for all}\ t \ge 0\ ,$$

$$(2.208)\qquad \|\partial_t\tilde{w}_1(t,x) - \partial_t\tilde{w}_2(t,x)\| \le MKd(\Phi_1,\Phi_2)(1+t)^{-\frac{m}{2}},\ \text{for all}\ t > 0\ ,$$

$$(2.209)\qquad \|\partial_t\partial_x\tilde{w}_1(t,x) - \partial_t\partial_x\tilde{w}_2(t,x)\| \le MKd(\Phi_1,\Phi_2)(t^{-\frac{1}{2}}+t^{-\frac{\alpha}{2}})(1+t)^{-\frac{m}{2}},$$

$$\text{for all}\ t > 0\ ,$$

where M is a constant independent of K, Φ_1, Φ_2 and t.

__Proof__. Denote by $\tilde{J}_1(t,x)$, $\tilde{J}_2(t,x)$ the first and second integral on the right-hand side of (2.202), respectively. We can prove above inequalities by the same procedure as in Lemmas 2.3, 2.5, and hence, it suffices to provide estimates for essential objects which occur in the process of proof. For $\tilde{J}_1(t,x)$, we need:

(2.210) $$\|\{p_u(w_{1\varepsilon}+z_{1\varepsilon},\theta_{1\varepsilon}) + a\}\partial_x w_{1\varepsilon} - \{p_u(w_{2\varepsilon}+z_{2\varepsilon},\theta_{2\varepsilon}) + a\}\partial_x w_{2\varepsilon}\|$$

$$\leq M\|\partial_x w_1\|\{\|\partial_x w_1-\partial_x w_2\| + \|\partial_x z_1-\partial_x z_2\| + \|\partial_x \theta_1-\partial_x \theta_2\|\}$$

$$+ M\{\|\partial_x w_2\| + \|\partial_x z_2\| + \|\partial_x \theta_2\|\}\|\partial_x w_1-\partial_x w_2\|$$

$$\leq MKd(\Phi_1,\Phi_2)(1+t)^{\frac{-1-m}{2}}, \quad \text{for all } t > 0 ,$$

where $w_{i\varepsilon} = w_i*\rho_\varepsilon$, $z_{i\varepsilon} = z_i*\rho_\varepsilon$, $\theta_{i\varepsilon} = \theta_i*\rho_\varepsilon$, $i = 1,2$,

(2.211) $$\|\tilde{M}_{qrs}(t,x)\| \leq (q+r+s+1)^2 MK^{q+r+s} d(\Phi_1,\Phi_2)(1+t)^{-\frac{m}{2}}, \text{ for all } t > 0 ,$$

(2.212) $$\|\partial_x \tilde{M}_{qrs}(t,x)\| \leq (q+r+s+1)^3 MK^{q+r+s} d(\Phi_1,\Phi_2)(t^{-\frac{1}{2}}+t^{-\frac{\alpha}{2}})(1+t)^{-\frac{m}{2}},$$
$$\text{for all } t > 0 ,$$

where $\tilde{M}_{qrs}(t,x) \overset{\text{def}}{=} \partial_t \int_{\frac{t}{2}}^{t} G_{12}(t-\tau,x)*\{(w_1^{q+1})_x z_1^r\theta_1^s - (w_2^{q+1})_x z_2^r\theta_2^s\}(\tau,x)d\tau$. For $\tilde{J}_2(t,x)$, we need:

(2.213) $$\|\{p(w_1+z_1,\theta_1) + aw_1 + az_1 + b\theta_1\} - \{p(w_2+z_2,\theta_2) + aw_2 + az_2 + b\theta_2\}\|$$

$$\leq MKd(\Phi_1,\Phi_2)(1+t)^{-\frac{1}{2}}, \quad \text{for all } t > 0 .$$

(2.214) $$\|\partial_x\{p(w_1+z_1,\theta_1) + aw_1 + az_1 + b\theta_1\} - \partial_x\{p(w_2+z_2,\theta_2) + aw_2 + az_2 + b\theta_2\}\|$$
$$\leq MKd(\Phi_1,\Phi_2)(1+t)^{-1}, \quad \text{for all } t > 0 .$$

Lemma 2.24. It holds that

$$\|\tilde{z}_1(t,x) - \tilde{z}_2(t,x)\| \le MKd(\Phi_1,\Phi_2), \quad \text{for all} \quad t \ge 0 \quad , \tag{2.215}$$

$$\|\partial_x\tilde{z}_1(t,x) - \partial_x\tilde{z}_2(t,x)\| \le MKd(\Phi_1,\Phi_2)(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t \ge 0 \quad , \tag{2.216}$$

$$\|\partial_{xx}\tilde{z}_1(t,x) - \partial_{xx}\tilde{z}_2(t,x)\| \le MKd(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \text{ for all } t > 0 \ . \tag{2.217}$$

Proof. Let us denote the five integrals of (2.203) by $\tilde{J}_3(t,x)$, $\tilde{J}_4(t,x)$, $\tilde{J}_5(t,x)$, $\tilde{J}_6(t,x)$ and $\tilde{J}_7(t,x)$ in sequence. To get the above estimates, we go through the same process as in Lemmas 2.7 to 2.11 with the following estimates. For $\tilde{J}_3(t,x)$, we need:

$$\begin{aligned}&\|\{p_u(w_1+z_1,\theta_1)+a\}\partial_x z_1 + \{p_\theta(w_1+z_1,\theta_1)+b\}\partial_x\theta_1 - \{p_u(w_2+z_2,\theta_2)+a\}\partial_x z_2 \\ &\quad - \{p_\theta(w_2+z_2,\theta_2) + b\}\partial_x\theta_2\| \le MKd(\Phi_1,\Phi_2)(1+t)^{-1}, \quad \text{for all} \quad t > 0 \quad .\end{aligned} \tag{2.218}$$

$$\begin{aligned}&\|\partial_x[\{p_u(w_1+z_1,\theta_1)+a\}\partial_x z_1 + \{p_\theta(w_1+z_1,\theta_1)+b\}\partial_x\theta_1 - \{p_u(w_2+z_2,\theta_2)+a\}\partial_x z_2 \\ &\quad - \{p_\theta(w_2+z_2,\theta_2)+b\}\partial_x\theta_2]\| \le MKd(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{\frac{-1-\alpha}{2}}, \text{ for all } \quad t > 0 \ .\end{aligned} \tag{2.219}$$

For $\tilde{J}_4(t,x)$, we use (2.213) and (2.214). For $\tilde{J}_5(t,x)$, we need:

$$\left\|\int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{(w_1+z_1)\partial_x v_1 - (w_2+z_2)\partial_x v_2\}(\tau,x)d\tau\right\| \le MKd(\Phi_1,\Phi_2) \quad , \tag{2.220}$$

for all $t \ge 0$, which can be obtained like the proof of (2.59).

$$\begin{aligned}&\left\|\int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\theta_1\partial_x v_1 - \theta_2\partial_x v_2\}(\tau,x)d\tau\right\| \le \\ &\quad \le \left\|\int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\frac{1}{d}\{\theta_1\partial_\tau\theta_1 - \theta_2\partial_\tau\theta_2\}(\tau,x)d\tau\right\| \\ &\quad + \left\|\int_0^{\frac{t}{2}} G_{13}(t-\tau,x)*\{\theta_1(\partial_x v_1 - \frac{1}{d}\partial_\tau\theta_1) - \theta_2(\partial_x v_2 - \frac{1}{d}\partial_\tau\theta_2)\}(\tau,x)d\tau\right\|\end{aligned} \tag{2.221}$$

$$\leq MKd(\Phi_1,\Phi_2) \quad ,$$

which follows from

$$\sup_{t>0} t^{\frac{1}{2}}(1+t)^{\frac{\alpha}{2}}\|\partial_t\theta_1 - d\partial_x v_1 - \partial_t\theta_2 + d\partial_x v_2\| \leq d(\Phi_1,\Phi_2) \quad .$$

$$\|\sum_{2\leq q+r}^{\infty} a_{qr}(w_1+z_1)^q\theta_1^r\partial_x v_1 - \sum_{2\leq q+r}^{\infty} a_{qr}(w_2+z_2)^q\theta_2^r\partial_x v_2\|$$

(2.222)

$$\leq MK^2 d(\Phi_1,\Phi_2)(1+t)^{-\frac{3}{2}}, \quad \text{for all} \quad t > 0 \quad .$$

$$\|\{\frac{p_\theta(w_1+z_1,\theta_1)}{e_\theta(w_1+z_1,\theta_1)}(\bar{\theta}+\theta_1) + d\}\partial_x v_1 - \{\frac{p_\theta(w_2+z_2,\theta_2)}{e_\theta(w_2+z_2,\theta_2)}(\bar{\theta}+\theta_2) + d\}\partial_x v_2\|$$

(2.223)

$$\leq MKd(\Phi_1,\Phi_2)(1+t)^{-1}, \quad \text{for all} \quad t > 0 \quad .$$

$$\|\partial_x[\{\frac{p_\theta(w_1+z_1,\theta_1)}{e_\theta(w_1+z_1,\theta_1)}(\bar{\theta}+\theta_1) + d\}\partial_x v_1] - \partial_x[\{\frac{p_\theta(w_2+z_2,\theta_2)}{e_\theta(w_2+z_2,\theta_2)}(\bar{\theta}+\theta_2) + d\}\partial_x v_2]\|$$

(2.224)

$$\leq MKd(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{\frac{-1-\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad .$$

For $\bar{J}_6(t,x)$ and $\bar{J}_7(t,x)$, we need:

$$\|\frac{1}{e_\theta(w_1+z_1,\theta_1)}(\partial_x v_1)^2 - \frac{1}{e_\theta(w_2+z_2,\theta_2)}(\partial_x v_2)^2\| \leq MKd(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{\frac{-1-\alpha}{2}},$$

(2.225)

$$\text{for all} \quad t > 0 \quad .$$

$$\left\| \partial_x\left\{\frac{1}{e_\theta(w_1+z_1,\theta_1)}(\partial_x v_1)^2\right\} - \partial_x\left\{\frac{1}{e_\theta(w_2+z_2,\theta_2)}(\partial_x v_2)^2\right\}\right\| \tag{2.226}$$

$$\leq MKd(\Phi_1,\Phi_2)t^{-1}(1+t)^{-\alpha}, \quad \text{for all} \quad t > 0 \quad .$$

$$\left\| \left\{\frac{1}{e_\theta(w_1+z_1,\theta_1)} - c\right\}\partial_{xx}\theta_1 - \left\{\frac{1}{e_\theta(w_2+z_2,\theta_2)} - c\right\}\partial_{xx}\theta_2 \right\| \tag{2.227}$$

$$\leq MKd(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{\frac{-1-\alpha}{2}}, \quad \text{for all} \quad t > 0 \quad .$$

<u>Lemma 2.25</u>. It holds that

$$\|\tilde{v}_1(t,x) - \tilde{v}_2(t,x)\| \leq MKd(\Phi_1,\Phi_2), \quad \text{for all} \quad t \geq 0 \quad , \tag{2.228}$$

$$\|\partial_x\tilde{v}_1(t,x) - \partial_x\tilde{v}_2(t,x)\| \leq MKd(\Phi_1,\Phi_2)(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad , \tag{2.229}$$

$$\|\partial_{xx}\tilde{v}_1(t,x) - \partial_{xx}\tilde{v}_2(t,x)\| \leq MKd(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \text{ for all } t > 0 \; . \tag{2.230}$$

<u>Proof</u>. Define

$$\tilde{Q}_{qrs}(t,x) = \partial_{xx}\int_{\frac{t}{2}}^{t} G_{22}(t-\tau,x)*\{\partial_x(w_1^q z_1^r \theta_1^s) - \partial_x(w_2^q z_2^r \theta_2^s)\}(\tau,x)d\tau \tag{2.231}$$

and

$$\tilde{R}_{qrs}(t,x) = \partial_{xx}\int_0^{\frac{t}{2}} G_{22}(t-\tau,x)*\{\partial_x(w_1^q z_1^r \theta_1^s) - \partial_x(w_2^q z_2^r \theta_2^s)\}(\tau,x)d\tau \quad . \tag{2.232}$$

Then, we have

$$\|\tilde{Q}_{qrs}(t,x)\| \leq (q+r+s)^2 MK^{q+r+s-1}d(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \tag{2.233}$$

$$\text{for all} \quad t > 0,\; q+r+s \geq 2,\; q \geq 1 \quad .$$

$$\|\tilde{Q}_{ors}(t,x)\| \leq (r+s)^2(r+s-1)MK^{r+s-1}d(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \tag{2.234}$$

$$\text{for all} \quad t > 0,\; r+s \geq 2.$$

$$\|\tilde{R}_{qrs}(t,x)\| \leq (q+r+s)^2 MK^{q+r+s-1}d(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \tag{2.235}$$

$$\text{for all} \quad t > 0,\; q+r+s \geq 2 \quad .$$

These inequalities combined with (2.213), (2.214), (2.222) to (227) and the inequalities analogous to (2.220), (2.221) will yield (2.228), (2.229) and (2.230) by the same procedure as in Lemmas 2.12, 2.14.

Lemma 2.26. It holds that

$$\|\tilde{\theta}_1(t,x) - \tilde{\theta}_2(t,x)\| \le MKd(\phi_1,\phi_2), \quad \text{for all } t > 0 , \tag{2.236}$$

$$\|\partial_x\tilde{\theta}_1(t,x) - \partial_x\tilde{\theta}_2(t,x)\| \le MKd(\phi_1,\phi_2)(1+t)^{-\frac{1}{2}}, \quad \text{for all } t > 0 , \tag{2.237}$$

$$\|\partial_{xx}\tilde{\theta}_1(t,x) - \partial_{xx}\tilde{\theta}_2(t,x)\| \le MKd(\phi_1,\phi_2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}} \quad \text{for all } t > 0 , \tag{2.238}$$

$$|||\partial_{xx}\tilde{\theta}_1(t,x) - \partial_{xx}\tilde{\theta}_2(t,x)|||_\alpha \le MKd(\phi_1,\phi_2)t^{\frac{-1-\alpha}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all } t > 0 . \tag{2.239}$$

Proof. In addition to the inequalities used in the proof of Lemma 2.25, we need only the following inequality:

$$\begin{aligned} &|||\{\frac{1}{e_\theta(w_1+z_1,\theta_1)} - c\}\partial_{xx}\theta_1(t,x) - \{\frac{1}{e_\theta(w_2+z_2,\theta_2)} - c\}\partial_{xx}\theta_2(t,x)|||_\alpha \le \\ &\le MKd(\phi_1,\phi_2)t^{\frac{-1-\alpha}{2}}(1+t)^{\frac{-1-\alpha}{2}}, \quad \text{for all } t > 0 , \end{aligned} \tag{2.240}$$

which is easily seen from Lemma 2.19. Repetition of the arguments in the proof of Lemmas 2.15 to 2.20 gives (2.236) to (2.239).

Lemma 2.27. It holds that

$$\|\partial_t\tilde{z}_1(t,x) - \partial_t\tilde{z}_2(t,x)\| \le MKd(\phi_1,\phi_2)(1+t)^{-\frac{1}{2}}, \quad \text{for all } t > 0 , \tag{2.241}$$

$$\|\partial_t\tilde{v}_1(t,x) - \partial_t\tilde{v}_2(t,x)\| \le MKd(\phi_1,\phi_2)t^{-\frac{1}{2}}, \quad \text{for all } t > 0 , \tag{2.242}$$

$$\|\partial_t\tilde{\theta}_1(t,x) - \partial_t\tilde{\theta}_2(t,x)\| \le MKd(\phi_1,\phi_2)t^{-\frac{1}{2}}, \quad \text{for all } t > 0 , \tag{2.243}$$

$$(2.244)\quad \|\partial_t\tilde{\theta}_1(t,x) - d\partial_x\tilde{v}_1(t,x) - \partial_t\tilde{\theta}_2(t,x) + d\partial_x\tilde{v}_2(t,x)\|$$

$$\leq MKd(\Phi_1,\Phi_2)t^{-\frac{1}{2}}(1+t)^{-\frac{\alpha}{2}}, \quad \text{for all} \quad t > 0 \ .$$

<u>Proof</u>. The assertions follow immediately from the above lemmas and the equations:

$$(2.245)\begin{cases} (\tilde{w}_1+\tilde{z}_1-\tilde{w}_2-\tilde{z}_2)_t = (\tilde{v}_1-\tilde{v}_2)_x \\ (\tilde{v}_1-\tilde{v}_2)_t = a(\tilde{w}_1+\tilde{z}_1-\tilde{w}_2-\tilde{z}_2)_x + b(\tilde{\theta}_1-\tilde{\theta}_2)_x + (\tilde{v}_1-\tilde{v}_2)_{xx} \\ \quad - \{p(w_1+z_1,\theta_1) + aw_1 + az_1 + b\theta_1 - p(w_2+z_2,\theta_2) - aw_2 - az_2 - b\theta_2\}_x \\ (\tilde{\theta}_1-\tilde{\theta}_2)_t = d(\tilde{v}_1-\tilde{v}_2)_x + c(\tilde{\theta}_1-\tilde{\theta}_2)_{xx} - \{\frac{p_\theta(w_1+z_1,\theta_1)}{e_\theta(w_1+z_1,\theta_1)}(\bar{\theta}+\theta_1) + d\}\partial_x v_1 \end{cases}$$

$$+ \{\frac{p_\theta(w_2+z_2,\theta_2)}{e_\theta(w_2+z_2,\theta_2)}(\bar{\theta}+\theta_2) + d\}\partial_x v_2 + \frac{1}{e_\theta(w_1+z_1,\theta_1)}(\partial_x v_1)^2$$

$$- \frac{1}{e_\theta(w_2+z_2,\theta_2)}(\partial_x v_2)^2 + \{\frac{1}{e_\theta(w_1+z_1,\theta_1)} - c\}\partial_{xx}\theta_1 - \{\frac{1}{e_\theta(w_2+z_2,\theta_2)} - c\}\partial_{xx}\theta_2 \ .$$

From Lemmas 2.23 to 2.27, we deduce:

<u>Proposition 2.28</u>. T is a contraction if

$$(2.246)\qquad M_4K < 1 \ ,$$

where M_4 is the sum of all M which appear in Lemmas 2.23 to 2.27 plus three times M in (2.209).

Now we are in a position to conclude the proof of our main theorem. First choose K, such that (2.19), (2.201), (2.246) and

$$(2.247)\qquad 0 < K < \min(\bar{\theta},\bar{u})$$

hold. Then T is a contraction from χ into itself if $\mu > 0$ is so small that (2.200) holds, and the unique fixed point of T is a solution of (0.11), (0.7) by setting $u = w+z$, which is easily seen from (2.30). Our proof is completed by the following lemma which implies that this solution is also a

solution to (0.6).

<u>Lemma 2.29</u>. Let (u,v,θ) be the solution mentioned above. Then,

$$\partial_t e(u,\theta) = e_u(u,\theta)\partial_t u + e_\theta(u.\theta)\partial_t\theta \quad , \tag{2.248}$$

$$\partial_t(\tfrac{1}{2} v^2) = v\partial_t v = -v\partial_x p(u,\theta) + v\partial_{xx}v \quad , \tag{2.249}$$

$$\partial_x\{vp(u,\theta)\} = (\partial_x v)p(u,\theta) + v\partial_x p(u,\theta) \quad , \tag{2.250}$$

$$\partial_x(v\partial_x v) = (\partial_x v)^2 + v\partial_{xx}v \quad , \tag{2.251}$$

$$e_u(u,\theta)\partial_t u = \{(\bar\theta+\theta)p_\theta(u,\theta) - p(u,\theta)\}\partial_x v \tag{2.252}$$

hold in $\mathcal{D}^*((0,\infty) \times R)$.

<u>Proof</u>. First of all, we note that $e_u(u,\theta)$, $e_\theta(u,\theta) \in C((0,\infty); L^\infty)$, $p(u,\theta) \in C((0,\infty); L^1 \cap BV)$ and $v,\theta \in C((0,\infty); C_0)$, which follow from the properties of χ and the fact that $W^{1,1} \subset C_0$. Suppose ε is any given positive number and define

$$\tilde u_\delta(t,x) = \int_\varepsilon^\infty \rho_\delta(t-\tau)u(\tau,x)*\rho_\delta(x)d\tau \quad ,$$

$$\tilde v_\delta(t,x) = \int_\varepsilon^\infty \rho_\delta(t-\tau)v(\tau,x)*\rho_\delta(x)d\tau \quad ,$$

$$\tilde \theta_\delta(t,x) = \int_\varepsilon^\infty \rho_\delta(t-\tau)\theta(\tau,x)*\rho_\delta(x)d\tau \quad ,$$

where $0 < \delta \leq \varepsilon$. Then, we see that

$$\tilde u_\delta(t,x) \in C^\infty(R^2) \cap C^1(R; L^1 \cap BV) \quad ,$$
$$\tilde v_\delta(t,x) \in C^\infty(R^2) \cap C(R; W^{1,1}) \cap C^1(R; M) \quad ,$$
$$\tilde \theta_\delta(t,x) \in C^\infty(R^2) \cap C(R; W^{1,1}) \cap C^1(R; L^1) \quad ,$$

and

$$\tilde u_\delta(t,x) \to u(t,x) \text{ in } L^1,\ \partial_x\tilde u_\delta(t,x) \to \partial_x u(t,x) \text{ weak * in } M \quad ,$$
$$\partial_t\tilde u_\delta(t,x) \to \partial_t u(t,x) \text{ in } L^1,\ \tilde v_\delta(t,x) \to v(t,x) \text{ in } C_0 \cap W^{1,1} \quad ,$$
$$\partial_t\tilde v_\delta(t,x) \to \partial_t v(t,x),\ \partial_{xx}\tilde v_\delta(t,x) \to \partial_{xx}v(t,x) \text{ weak * in } M \quad ,$$
$$\tilde\theta_\delta(t,x) \to \theta(t,x) \text{ in } C_0 \cap W^{1,1},\ \partial_t\tilde\theta_\delta(t,x) \to \partial_t\theta(t,x) \text{ in } L^1 \quad ,$$

for each $t \in [2\varepsilon,\infty)$ as $\delta \to 0$. Hence, it holds that

$$\|e(\tilde u_\delta,\tilde\theta_\delta) - e(u,\theta)\| \to 0 \quad ,$$

$$e_u(\tilde{u}_\delta,\tilde{\theta}_\delta)\partial_t\tilde{u}_\delta + e_\theta(\tilde{u}_\delta,\tilde{\theta}_\delta)\partial_t\tilde{\theta}_\delta \to e_u(u,\theta)\partial_t u + e_\theta(u,\theta)\partial_t\theta \quad \text{in} \quad L^1 ,$$

$$\tilde{v}_\delta^2 \to v^2 \quad \text{in} \quad L^1, \ \tilde{v}_\delta\partial_t\tilde{v}_\delta \to v\partial_t v \quad \text{weak * in} \quad M ,$$

$$\tilde{v}_\delta p(\tilde{u}_\delta,\tilde{\theta}_\delta) \to vp(u,\theta), \ (\partial_x\tilde{v}_\delta)p(\tilde{u}_\delta,\tilde{\theta}_\delta) \to (\partial_x v)p(u,\theta) \quad \text{in} \quad L^1 ,$$

$$\tilde{v}_\delta\partial_x p(\tilde{u}_\delta,\tilde{\theta}_\delta) \to v\partial_x p(u,\theta) \quad \text{weak * in} \quad M, \ \tilde{v}_\delta\partial_x\tilde{v}_\delta \to v\partial_x v \quad \text{in} \quad L^1 ,$$

$$(\partial_x\tilde{v}_\delta)^2 \to (\partial_x v)^2 \quad \text{in} \quad L^1, \ \tilde{v}_\delta\partial_{xx}\tilde{v}_\delta \to v\partial_{xx}v \quad \text{weak * in} \quad M ,$$

for each $t \in [2\varepsilon,\infty)$, from which (2.248), (2.250), (2.251) and the first part of (2.249) follow, since ε was arbitrarily chosen. Using the fact that $v \in C((0,\infty); C_0)$, $-\partial_x p(u,\theta) + \partial_{xx}v \in C((0,\infty); M)$ the second part of (2.249) follows from the equation:

$$\partial_t v = -\partial_x p(u,\theta) + \partial_{xx}v \quad \text{in} \quad \mathcal{D}^*((0,\infty) \times R) \quad .$$

Finally, (2.252) is an immediate consequence of (0.9).

Remark 2.30. It has not been proved that the solution we obtained above is unique, which is still open. However, the solution has an interesting feature: if the initial data have jump discontinuities, then the discontinuities of v, θ vanish instantaneously while the strength of jump discontinuity of u vanishes at least as fast as the inverse of a polynomial.

APPENDIX

[A1] We shall prove that the expression (2.36) is valid. First note that $w \in C^1((0,\infty); L^1 \cap BV)$, $z \in C^1((0,\infty); L^1 \cap BV) \cap C((0,\infty); W^{1,1})$, $\theta \in C^1((0,\infty); L^1) \cap C((0,\infty); W^{1,1})$, from which we have

$$(w^{q+1})_x z^r \theta^s = -w^{q+1}(z^r\theta^s)_x + (w^{q+1}z^r\theta^s)_x \in C((0,\infty); M)$$

and

$$w^{q+1}z^r\theta^s \in C^1((0,\infty); L^1) \quad .$$

Next we define

$$N_{1,\varepsilon}(t,x) = -\int_{\frac{t}{2}}^{\max(t-\varepsilon,\frac{t}{2})} G_{12}(t-\tau,x)*\{w^{q+1}(z^r\theta^s)_x\}(\tau,x)d\tau \quad ,$$

$$N_{2,\varepsilon}(t,x) = \int_{\frac{t}{2}}^{\max(t-\varepsilon,\frac{t}{2})} G_{12}(t-\tau,x)*(w^{q+1}z^r\theta^s)(\tau,x)d\tau \quad .$$

Then $N_{1,\varepsilon}(t,x)$, $N_{2,\varepsilon}(t,x)$ are well-defined and $\partial_t N_{1,\varepsilon}(t,x) + \partial_t\partial_x N_{2,\varepsilon}(t,x) \to M_{qrs}(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$. Since $\partial_t G_{12}(t,x) = \partial_x G_{22}(t,x)$ in $\mathcal{D}^*((0,\infty) \times R)$, $G_{12}(t,x) \in C^1((0,\infty); L^1)$. At the same time, we see that $G_{12}(t,x) \in C([0,\infty); L^1)$ with $G_{12}(0,x) = 0$, $\partial_x G_{12}(t,x) = H_5(t,x) - e^{-at}\delta(x)$, and $H_5(t,x) \in C((0,\infty); L^1)$ with $\|H_5(t,x)\| \leq M(t^{\frac{1}{2}} + t^{\frac{1}{6}})^{-1}$, for all $t > 0$. Now we can compute $\partial_t N_{1,\varepsilon}(t,x)$ and $\partial_t\partial_x N_{1,\varepsilon}(t,x)$ by integration by parts which is valid from the properties stated above. Then, letting $\varepsilon \to 0$, we obtain the result.

[A2] We shall prove that $(\tilde{w},\tilde{z},\tilde{v},\tilde{\theta})$ defined by (2.26) to (2.29) satisfies (2.30) in $\mathcal{D}^*((0,\infty) \times R)$. (2.30) can be written in the form with different notations,

$$(3.1)\qquad \begin{cases} u_t = v_x \\ v_t = au_x + b\theta_x + v_{xx} + f_1(t,x) \\ \theta_t = dv_x + c\theta_{xx} + f_2(t,x) \quad , \end{cases}$$

where

$$(3.2)\qquad \begin{cases} f_1(t,x) \in C((0,\infty); \mathcal{M}) \\ \|f_1(t,x)\| \le M(1+t)^{-1}, \quad \text{for all} \quad t > 0 \quad , \end{cases}$$

$$(3.3)\qquad \begin{cases} f_2(t,x) \in C((0,\infty); L^1) \\ \|f_2(t,x)\| \le Mt^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \quad . \end{cases}$$

Applying the Fourier transform, (3.1) with given initial data yields

$$(3.4)\qquad \frac{\partial}{\partial t}\hat{Y}(t,\xi) = \hat{A}(\xi)\hat{Y}(t,\xi) + \hat{F}(t,\xi) \quad ,$$

$$(3.5)\qquad \hat{Y}(0,\xi) = \begin{pmatrix} \hat{u}_0(\xi) \\ \hat{v}_0(\xi) \\ \hat{\theta}_0(\xi) \end{pmatrix} \quad ,$$

where $\hat{F}(t,\xi) = \begin{pmatrix} 0 \\ \hat{f}_1(t,\xi) \\ \hat{f}_2(t,\xi) \end{pmatrix}$ and $\hat{A}(\xi)$, $\hat{Y}(t,\xi)$ are given by (1.2). From

(3.2), (3.3), it follows that

$$(3.2)^* \qquad \begin{cases} \hat{f}_1(t,\xi) \in C((0,\infty);\ C(R) \cap L^\infty) \\ \|\hat{f}_1(t,\xi)\|_{L^\infty} \leq M(1+t)^{-1}, \quad \text{for all} \quad t > 0 \ , \end{cases}$$

$$(3.3)^* \qquad \begin{cases} \hat{f}_2(t,\xi) \in C((0,\infty);\ C_0(R)) \\ \|\hat{f}_2(t,\xi)\|_{L^\infty} \leq Mt^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}, \quad \text{for all} \quad t > 0 \ . \end{cases}$$

Since u_0, v_0, $\theta_0 \in L^1 \cap BV$, we have $\hat{u}_0(\xi)$, $\hat{v}_0(\xi)$, $\hat{\theta}_0(\xi) \in C_0(R)$. Now for each $\xi \in R$, the unique solution to (3.4), (3.5) is given by

$$(3.6) \qquad \hat{Y}(t,\xi) = \hat{G}(t,\xi)\hat{Y}(0,\xi) + \int_0^t \hat{G}(t-\tau,\xi)\hat{F}(\tau,\xi)d\tau \ .$$

We recall that $\hat{G}_{ij}(t,\xi) \in C((0,\infty);\ C(R) \cap L^\infty)$ and $\|\hat{G}_{ij}(t,\xi)\|_{L^\infty} \leq M$, for all $t > 0$, $i,j = 1,2,3$. Hence, it is obvious that $\hat{Y}(t,\xi)$ given by (3.6) satisfies

$$-\int_{-\infty}^{\infty}\int_0^\infty \hat{Y}(t,\xi)\phi_t(t)\psi(\xi)dtd\xi = \int_{-\infty}^{\infty}\int_0^\infty \hat{A}(\xi)\hat{Y}(t,\xi)\phi(t)\psi(\xi)dtd\xi$$

$$+\int_{-\infty}^{\infty}\int_0^\infty \hat{F}(t,\xi)\phi(t)\psi(\xi)dtd\xi \ ,$$

for all $\phi \in C_0^\infty((0,\infty))$ and ψ of the Schwartz space in R. Therefore $\mathcal{F}_\xi^{-1}\hat{Y}(t,\xi)$ satisfies (3.1) in $\mathcal{D}^*((0,\infty) \times R)$. But $\mathcal{F}_\xi^{-1}\hat{Y}(t,\xi)$ is precisely $(\tilde{w}+\tilde{z},\tilde{v},\tilde{\theta})$ given by (2.26) to (2.29).

REFERENCES

[1] Ahlfors, L., "Complex Analysis," 2nd ed., McGraw-Hill, New York, 1966.

[2] Dafermos, C., "Conservation Laws with Dissipation" in "Nonlinear Phenomena in Mathematical Sciences" ed. by U. Lakshimikantham, Academic Press.

[3] Dafermos, C., "Global Smooth Solutions to the Initial-Boundary Value Problem for the Equations of One-dimensional Nonlinear Thermoviscoelasticity", SIAM J. of Math. Analyis (to appear).

[4] Dafermos, C. and Hsiao, L., "Global Smooth Thermomechanical Processes in One-dimensional Nonlinear Thermoviscoelasticity", Nonlinear Analysis (to appear).

[5] Liu, T. P., "Solutions in the Large for Equations of Non-isentropic Gas Dynamics," Indiana Univ. Math. J. 26 (1977), pp. 137-168.

[6] Kim, J., "Solutions to the Equations of One Dimensional Viscoelasticity in BV," LCDS Report 81-13, Brown University (1981).

[7] Slemrod, M., "Global Existence, Uniqueness and Asymptotic Stability of Classical Smooth Solutions in One-dimensional Non-linear Thermoelasticity," Arch. Rat. Mech. Anal. 76 (1981), pp. 97-134.

[8] Stein, E., "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, 1970.

JUK/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER # 2364	2. GOVT ACCESSION NO. AD-A116324	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Global Existence of Solutions of the Equations of One-Dimensional Thermoviscoelasticity with Initial Data in BV and L^1		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Jong Uhn Kim		8. CONTRACT OR GRANT NUMBER(s) MCS-7927062 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS see Item 18 below		12. REPORT DATE April 1982
		13. NUMBER OF PAGES 81
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Equations of one-dimensional nonlinear thermoviscoelasticity, Linear equations, Functions of bounded variation (BV), Fourier transform, Global solutions, Variation of constants formula.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider the Cauchy problem associated with the equations:

$$(1)\begin{cases} u_t = v_x \\ v_t = -p(u,\theta)_x + v_{xx} \\ [e(u,\theta) + \frac{1}{2}v^2]_t + [p(u,\theta)v]_x - [vv_x]_x = \theta_{xx}, \; x \in R,\; t \in R^+ \end{cases},$$

with the initial condition

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

(continued)

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT (continued)

(2) $$u(0,x) = u_0(x), \ v(0,x) = v_0(x), \ \theta(0,x) = \theta_0(x) \quad .$$

The equations (1) describe the one-dimensional motion of a particular type of nonlinear thermoviscoelastic material. We establish the existence of global solutions when the initial data belong to $L^1 \cap BV$ and are sufficiently small in terms of $L^1 \cap BV$. Our method consists of linearization, Fourier transformations and contraction mapping principle via variation of constants formula.